Abel's theorem

Summation by parts: For any numbers a_k, b_k,

$$\sum_{k=1}^{n} a_k b_k = a_n(b_1 + \ldots + b_n) + \sum_{k=1}^{n-1}(a_k - a_{k+1})(b_1 + \ldots + b_k).$$

Proof: The formula clearly holds when $n = 1:$ $a_1 b_1 = a_1 b_1$. As we increment n by 1, the left-hand side increases by $a_{n+1}b_{n+1}$ and the right-hand side increases by

$$(a_{n+1} - a_n)(b_1 + \ldots + b_n) + a_{n+1}b_{n+1} + (a_n - a_{n+1})(b_1 + \ldots + b_n) = a_{n+1}b_{n+1}. \qquad \square$$

Abel's bound: If $a_k > 0$ are decreasing and b_k have bounded sums, $|b_1 + \ldots + b_k| \le B$,

$$\left|\sum_{k=1}^{n} a_k b_k\right| \le a_1 B.$$

Proof: Using summation by parts,

$$\left|\sum_{k=1}^{n} a_k b_k\right| \le \left(a_n + \sum_{k=1}^{n-1}(a_k - a_{k+1})\right)B = a_1 B. \qquad \square$$

Abel's test for uniform convergence: Suppose that $a_k(x)$ and $b_k(x)$ are two sequences of functions on a set E satisfying the following three conditions:

- $a_k(x)$ is a monotone sequence for every $x \in E$,
- $a_k(x)$ are uniformly bounded: $|a_k(x)| \le A$, $k \in \mathbb{N}$, $x \in E$,
- $\sum_{k=1}^{\infty} b_k(x)$ converges uniformly on E.

Then the series $\sum_{k=1}^{\infty} a_k(x) b_k(x)$ converges uniformly on E.

Proof: We may assume that $a_k(x)$ are decreasing. Indeed, for any $x \in E$ such that $a_k(x)$ is increasing, replace $a_k(x)$ and $b_k(x)$ with $-a_k(x)$ and $-b_k(x)$, respectively. We may also assume that $a_k(x) \ge 0$ by considering $a_k(x) + A$ in place of $a_k(x)$.
Using Abel's bound, we have

$$\left|\sum_{k=i}^{j} a_k(x) b_k(x)\right| \le a_i(x) \sup_{m \ge i}\left|\sum_{k=i}^{m} b_k(x)\right| \le A \sup_{m \ge i}\left|\sum_{k=i}^{m} b_k(x)\right| \rightrightarrows 0, \qquad i \to \infty. \qquad \square$$

Theorem [Niels H. Abel, 1826]: If a power series $\sum_{k=0}^{\infty} c_k x^k$ converges at some $x_0 > 0$, it then converges *uniformly* on $[0,\ x_0]$. In particular, the series is left-continuous at x_0.

Proof: Apply Abel's convergence test with $a_k(x) = (x/x_0)^k$ and $b_k(x) = c_k x_0^k$. $\square$