23. ⟨3⟩ Prove that $a_n = \mathrm{Ord}(b_n)$ is an equivalence relation as defined on p. 15. If we now reinterpret $\mathrm{Ord}(a_n)$ to denote the equivalence class of $\{a_n\}$, it follows that the statement $\mathrm{Ord}(a_n) = \mathrm{Ord}(b_n)$ now makes sense as an equality of sets. See [25, Section 0.1]. *Note*: Using Condition (3), this equivalence relation is defined on the set of all sequences. Using (2) it must be restricted to sequences with only finitely many zero values, so that a_n/b_n is eventually defined.

24. ⟨2⟩ Since we used $<$ as the symbol for order ranking, it better be true that order ranking behaves the way $<$ should. Specifically, show that if $\mathrm{Ord}(a_n) < \mathrm{Ord}(b_n)$ and $\mathrm{Ord}(b_n) < \mathrm{Ord}(c_n)$ then $\mathrm{Ord}(a_n) < \mathrm{Ord}(c_n)$. That is, prove that order ranking is transitive.

25. ⟨3⟩ Is big Oh a transitive relation? Is it an equivalence relation?

26. ⟨3⟩ Define sequence $\{a_n\}$ to be **asymptotic** to $\{b_n\}$, written $a_n \sim b_n$, if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

If $\{a_n\}$ is not asymptotic to $\{b_n\}$, we write $a_n \not\sim b_n$.

a) Show that $3n-1 \sim 3n$ but $3n-1 \not\sim n$.

b) If $a_n \sim b_n$, then $a_n = \mathrm{Ord}(b_n)$. Why?

c) Show that $\sim$ is an equivalence relation.

d) Prove: If $a_n \sim b_n$ and $b_n = o(c_n)$, then $a_n = o(c_n)$.

e) Prove: If $a_n \sim b_n$ and $b_n = \mathrm{Ord}(c_n)$, then $a_n = \mathrm{Ord}(c_n)$.

27. ⟨3⟩

a) Prove that $\mathrm{Ord}(\log n) = \mathrm{Ord}(\lceil \log n \rceil)$.

b) More generally, for any function $f(n)$ show that $\mathrm{Ord}(f(n)\lfloor \log n \rfloor) = \mathrm{Ord}(f(n)\log n) = \mathrm{Ord}(f(n)\lceil \log n \rceil)$. For instance, it follows that $\mathrm{Ord}(n^k \log n) = \mathrm{Ord}(n^k \lceil \log n \rceil)$ for every power k.

This problem is relevant because algorithms must take an integer number of steps. Thus, when we say an algorithm takes $5n\log n$ steps, we probably mean something like $5n\lceil \log n \rceil$ steps. This problem suggests that we won't get the Order wrong by ignoring floors and ceilings.

28. ⟨4⟩ Ord, O, and o need not be restricted to sequences, or to limits at ∞. Consider the following version: Let f and g be functions defined around 0, and write $f(x) = \mathrm{Ord}(g(x))$ (as $x \to 0$) if there are positive constants L and U such that $L \le |f(x)/g(x)| \le U$ for all x sufficiently close to 0. While this cognate definition is not very useful for the analysis of algorithms, it can be quite useful in continuous mathematics.

a) Show that, as $x \to 0$, $\mathrm{Ord}(x+1) = \mathrm{Ord}(x+2)$ but that $\mathrm{Ord}(x+1) \neq \mathrm{Ord}(x)$.

b) Come up with the associated definition for $\mathrm{Ord}(f(x)) < \mathrm{Ord}(g(x))$.

c) Consider all functions of either the form $f(x) = x^n$, $n > 0$, or the form $f(x) = a^x$, $a > 0$. Determine the order ranking (as $x \to 0$) for all these functions.

0.4 Summation and Product Notation

In Section 0.1 we wrote a polynomial $P_n(x)$ as

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0. \qquad (1)$$

Perhaps you find reading this as tedious as we found typing it on our word processor. In any case, some shorthand notation other than the use of ellipses for writing long sums like this would clearly be useful and would save a lot of space, if nothing else. **Summation notation** is such a shorthand. It is one of the most powerful bits of mathematical notation there is, and we'll use it often. Although summation notation is simple in concept, students often have trouble using and understanding it. Since it is so important and since its use involves considerable subtleties, we devote an entire section to it and its close relative, product notation.

Summation notation allows us, for example, to write the polynomial in Eq. (1) as

$$P_n(x) = \sum_{i=0}^{n} a_i x^i. \tag{2}$$

Instead of writing a sequence of terms, we have written a single general term ($a_i x^i$) using a new subscript, the letter i. We could just as easily have used j or k or l or even n, if we weren't already using it for another purpose in Eq. (2). The letter i is called a **dummy variable** in that, were we to write out the entire expression, this variable wouldn't appear at all. (Section 1.4 discusses a related use of this terminology in algorithms.) Using the dummy variable, we have, in effect, captured the explicit essence of the *pattern* in Eq. (1), which is only implicit using an ellipsis. If you substitute $i = 0$ into the pattern term, you get $a_0 x^0 = a_0$, since a nonzero number raised to the zero power is 1. If you substitute 1 for i, you get $a_1 x^1 = a_1 x$, the linear term. And so on, until substituting n for i you get $a_n x^n$.

It is probably already clear that the $\sum$ in front of the pattern term indicates that you are to add all the terms you have gotten by substituting into the pattern. ($\sum$ is the Greek capital letter Sigma, which corresponds to a Roman "S"; hence Sigma for Sum.) The line below the $\sum$ gives the lowest value of the dummy variable, whereas the line above indicates its highest value. (Sometimes, if we want to write a summation in a paragraph rather than setting it off on a line by itself, we write the upper and lower limits to the right of the $\sum$, as in $\sum_{i=1}^{8}$.) What we have called so far the dummy variable is more commonly called the **index of summation** (or just the index). The index takes on all integral values between the lowest and highest values.

If you understand our explanation of Eq. (2), you'll immediately see that it had two advantages over use of ellipsis:

- It is more compact.
- Whereas the pattern in the use of an ellipsis may be obvious to the writer, it will sometimes be less than obvious to the reader; summation notation gets rid of any possible ambiguity.

Note also the power of using subscripts in Eq. (2) where a_i stands for any of the values $a_1, a_2, \ldots, a_n$ and x^i stands for the associated power of x. Without such use of subscripts, summation notation would not be possible, for how could you possibly represent, say, $a, b, \ldots, z$ by a single symbol?

In general, suppose we wish to express

$$c_j + c_{j+1} + \cdots + c_{k-1} + c_k$$

with summation notation. We do this as

$$\sum_{i=j}^{k} c_i,$$

where each c_i is called a **term** or **summand** of the summation. This general example includes our polynomial example: just set $j = 0$, $k = n$, and $c_i = a_i x^i$.

Eq. (2) indicates that the index of summation can appear in the quantity being summed as a subscript, an exponent, or indeed, in any way you wish. Or it need not appear at all, as in

$$\sum_{i=1}^{n} 1,$$

which represents the sum of n 1's since, for each value of i, the quantity being summed is the constant 1.

EXAMPLE 1 Evaluate:

a) $\sum_{i=3}^{6} i^2$ **b)** $\sum_{p=0}^{5}(2p+3)$ **c)** $\sum_{i=1}^{4} i^{2i}$.

Solution For a) we get

$$3^2 + 4^2 + 5^2 + 6^2 = 9 + 16 + 25 + 36 = 86.$$

For b) we have

$$(0{+}3) + (2{+}3) + (4{+}3) + (6{+}3) + (8{+}3) + (10{+}3) = 3 + 5 + 7 + 9 + 11 + 13 = 48.$$

Note that the index of summation was p. Note also that, if we had left the index of summation as p but had written $2i+3$ instead of $2p+3$, i.e.,

$$\sum_{p=0}^{5}(2i+3),$$

then this sum would have called for the addition six times of $2i+3$ (with the result $12i+18$). Why? Because anything after (i.e., to the right of) the summation sign which does not include the index is a constant with respect to the summation.

For c), where the index of summation appears more than once under the summation sign, we get

$$\begin{aligned} 1^2 + 2^{2\cdot 2} + 3^{2\cdot 3} + 4^{2\cdot 4} &= 1^2 + 2^4 + 3^6 + 4^8 \\ &= 1 + 16 + 729 + 65{,}536 \\ &= 66{,}282. \ \blacksquare \end{aligned}$$

The set of values taken on by the index of summation is called the **index set**. There is no requirement that this set consist of a sequence of consecutive integral values, although this is by far the most common case. For example, we may write

$$\sum_{\substack{i=2 \\ i \text{ even}}}^{8} a_i, \tag{3}$$

which represents $a_2 + a_4 + a_6 + a_8$. Or we may write

$$\sum_{\substack{i=1 \\ i \text{ not divisible} \\ \text{by 2 or 3}}}^{11} 5^i, \tag{4}$$

which represents

$$5 + 5^5 + 5^7 + 5^{11} = 5 + 3125 + 78{,}125 + 48{,}828{,}125 = 48{,}909{,}380.$$

In full generality we may write

$$\sum_{R(i)} a_i, \tag{5}$$

where $R(i)$ is a function, often called a **predicate**, whose domain is usually N and which has the value *true* for each i included in the index set and the value *false* for all other i in the domain. (Are you disturbed at all by a function whose values are not numbers but rather *true* and *false*? No need to be. Remember that, like the domain, the codomain of a function may be any set whatsoever. In Chapter 7, when we discuss logic, we'll be dealing almost entirely with functions whose values are *true* or *false*.) For example, in part a) of Example 1, $R(i)$ would be true when $i = 3$, 4, 5, and 6 and false otherwise. In (3), $R(i)$ would be true only when $i = 2$, 4, 6, and 8. In (4), $R(i)$ would be true only when $i = 1$, 5, 7, and 11. Indeed, we could have written all these summations using predicate notation. For example, we could have written part a) of Example 1 as

$$\sum_{3 \le i \le 6} i^2,$$

although we wouldn't usually write it that way when the lower limit–upper limit notation is natural. By the way, we write the predicates *below* the summation sign merely because of convention.

We may, in fact, generalize (5) even further by not even requiring that the predicate refer to an integer index. For example, we may write

$$\sum_{x \in T} f(x),$$

where T is a set and f is a function defined on elements of T. Or we may write

$$\sum_{S \subset T} f(S),$$

where T is a set and f is a function defined on subsets of T.

One implication of (5) is that, if $R(i)$ is true for all nonnegative numbers, then that summation contains an *infinite* number of terms. We'll meet a few such summations in later chapters. The usual notation when summing over all of N is to put the infinity symbol ∞ over the Sigma, as in

$$\sum_{i=0}^{\infty} a_i.$$

Suppose $R(i)$ is not true for *any* value of i, as in

$$\sum_{8 \le i \le 6} i^3.$$

Then the sum is empty and, by convention, its value is zero. Or suppose we write

$$\sum_{i=8}^{6} a_i.$$

Since, again by convention, we *always* step up from the lower limit to the upper one, this sum is also empty and has the value zero.

The compactness of summation notation allows us to manipulate it easily in a variety of ways. For example, if c is a constant, then

$$\sum_{R(i)} ca_i \;=\; c\sum_{R(i)} a_i, \tag{6}$$

because the constant c multiplies every term and may therefore be factored out independently of $R(i)$. More generally, if c and d are constants, then

$$\sum_{R(i)} (ca_i + db_i) \;=\; c\sum_{R(i)} a_i \;+\; d\sum_{R(i)} b_i. \tag{7}$$

Rewriting summations. Throughout mathematics one must know how to **rewrite** expressions, that is, find alternative forms with the same value, for another form is often just what is needed to make progress. We now give three important examples of how to change the form of a summation.

First, we consider how to change the limits of the index of summation or, to put it another way, how to change the index of summation from one variable to another. A common case occurs when you have a sum of the form

$$\sum_{i=1}^{n} a_i \tag{8}$$

and you would like to change it so that the lower limit is 0. You do this by defining

$$j = i - 1 \qquad \text{that is,} \qquad i = j + 1. \tag{9}$$

When $i = 1$, $j = 0$ and when $i = n$, $j = n - 1$. Using these limits and replacing the subscript i by the equivalent $j + 1$, you may rewrite (8) as

$$\sum_{j=0}^{n-1} a_{j+1}. \tag{10}$$

If you write out (8) and (10), you will see that both have precisely the same terms, namely,

$$a_1 + a_2 + \cdots + a_n,$$

and thus they are equal. In actual practice, after we've made this change of variable, we often then go back to the original index and write

$$\sum_{i=0}^{n-1} a_{i+1}.$$

Remember that the index is a dummy variable and its "name", therefore, makes no difference.

With a change of variable like that in Eq. (9) you'll be able to change the limits of summation in almost any way you wish. You must remember, though, to change all places where the index of summation appears in the term under the summation sign.

Our second example concerns the case where you wish to reverse the order of summation. That is, when the summation is expanded, instead of having the terms appear in the natural order from lower limit to upper limit,

$$a_1 + a_2 + \cdots + a_n,$$

we wish to have them appear in the order

$$a_n + a_{n-1} + \cdots + a_2 + a_1.$$

We do this by making the change of variable

$$j = n - i,$$

which changes (8) to

$$\sum_{i=1}^{n} a_i = \sum_{j=0}^{n-1} a_{n-j}.$$

Our third example concerns a common occurrence where we have two separate summations that we would like to combine into one, e.g.,

$$\sum_{i=0}^{n} a_i + \sum_{i=1}^{n} b_i.$$

How can we combine these expressions, which have different limits, into a single summation? Remove the first term of the first summation to obtain

$$a_0 + \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i.$$

Now, since the limits on both summations are the same, we can write this as

$$a_0 + \sum_{i=1}^{n} (a_i + b_i).$$

Double Summation

Suppose we have a set of doubly subscripted quantities a_{ij}, where the range of values of i is m, $m{+}1$, ..., $n{-}1$, n and the range of j is p, $p{+}1$, ..., $q{-}1$, q. Now suppose we wish to add the a_{ij}'s for all possible pairs of values of i and j. We can do this with a double summation, which we write as

$$\sum_{i=m}^{n} \sum_{j=p}^{q} a_{ij} \tag{11}$$

and interpret as follows:

> For each value of the index in the outer (i.e., left-most) summation let the index in the inner summation range over all its values and sum the terms generated; then increase the outer index by 1 and repeat, always adding the result to the previous sum until i reaches n.

In effect, this means that (11) contains implied parentheses, as in

$$\sum_{i=m}^{n}\left(\sum_{j=p}^{q} a_{ij}\right).$$

Thus (11) represents the sum

$$\begin{aligned}\sum_{j=p}^{q} a_{mj} + \sum_{j=p}^{q} a_{m+1,j} + \cdots + \sum_{j=p}^{q} a_{nj} &= a_{mp} + a_{m,p+1} + a_{m,p+2} + \cdots \\ &+ a_{mq} + a_{m+1,p} + \cdots + a_{m+1,q} + \cdots + a_{np} + a_{n,p+1} + \cdots + a_{nq},\end{aligned} \tag{12}$$

which is what we wanted.

EXAMPLE 2 Evaluate:

$$\sum_{i=1}^{3}\sum_{j=2}^{4} i^j.$$

Solution We obtain

$$\begin{aligned}&(1^2 + 1^3 + 1^4) + (2^2 + 2^3 + 2^4) + (3^2 + 3^3 + 3^4) \\ &\qquad = 1 + 1 + 1 + 4 + 8 + 16 + 9 + 27 + 81 = 148.\end{aligned}$$ ∎

When the index values for the two sums are defined, again from left to right, by two general predicates $R(i)$ and $S(j)$, the principle is the same: For each value of i for which $R(i)$ is true, evaluate all the terms for which $S(j)$ is true and add them.

When a double sum is particularly simple, it can be written using a single sum. For instance,

$$\sum_{i,j=1}^{n} a_{ij} \quad \text{means the same as} \quad \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}.$$

Now suppose that the term under the summation sign in (11) is $a_i b_j$, that is, the product of two terms, one with subscript i and one with subscript j. Since Eq. (12) indicates that the double summation includes all possible combinations of the subscripts i and j, it follows that in this case we may write (11) as

$$\sum_{i=m}^{n}\sum_{j=p}^{q} a_i b_j = \left(\sum_{i=m}^{n} a_i\right)\left(\sum_{j=p}^{q} b_j\right), \tag{13}$$

since the product of the two sums includes all possible combinations of i and j. Another way to see this is to note that on the left of Eq. (13) a_i is a constant *with* *respect to the summation with index* j and therefore can be brought outside that summation as a constant, as in Eq. (6). For example, with $n = 2$, $q = 3$, and $m = p = 1$,

$$\begin{aligned}
\sum_{i=1}^{2}\sum_{j=1}^{3} a_i b_j &= \sum_{j=1}^{3} a_1 b_j + \sum_{j=1}^{3} a_2 b_j \\
&= a_1 \sum_{j=1}^{3} b_j + a_2 \sum_{j=1}^{3} b_j \qquad \text{[Constant brought out]} \\
&= (a_1 + a_2) \sum_{j=1}^{3} b_j \\
&= \Big(\sum_{i=1}^{2} a_i\Big)\Big(\sum_{j=1}^{3} b_j\Big).
\end{aligned}$$

If you understand this example, you should be able to justify Eq. (13) in general [16].

So far, then, double summation hasn't been much more difficult than single summation. And you should be able to extend the preceding discussion without any trouble to triple summation or to any number of sums in succession. However, suppose we write

$$\sum_{i=1}^{4}\sum_{j=1}^{5-i} \frac{j^2}{(2i-1)}. \tag{14}$$

Here one of the limits for the inner sum is not a constant but contains a variable, namely, the index of the outer sum. Still, it should be fairly clear that we should evaluate (14) as follows:

i) Set $i = 1$.
ii) Set $j = 1, 2, 3, 4$ (since $5 - i = 4$), evaluate $j^2/(2i - 1) = j^2$, and sum.
iii) Set $i = 2$ and $j = 1, 2, 3$ (since $5 - i = 3$), evaluate, and add to the result of (ii).
iv) Repeat for $i = 3$ and $j = 1, 2$.
v) Finally, repeat for $i = 4$; since here the upper and lower limits are the same, i.e., 1, we have only $j = 1$.

The result is

$$1 + 4 + 9 + 16 + \frac{1}{3} + \frac{4}{3} + \frac{9}{3} + \frac{1}{5} + \frac{4}{5} + \frac{1}{7} = 35\frac{17}{21}.$$

We could give other, even trickier examples, but all we want to do here is introduce you to summations with variable limits. From Eq. (12) it should be clear that, when all the limits are constants, (11) represents the same sum regardless of whether the i summation or the j summation comes first. Thus

$$\sum_{i=m}^{n} \sum_{j=p}^{q} a_{ij} = \sum_{j=p}^{q} \sum_{i=m}^{n} a_{ij}.$$

But what about summations like (14)? Clearly, we can't just interchange the order of summation, because then the outer summation would have a limit depending on a summation to its right, which doesn't make any sense because by definition double sums are expanded from left to right. We won't discuss just how you go about interchanging the order of summation when one or more of the limits is variable. But the need to do so does occur occasionally. Be on your guard if it ever does occur, because care is required to handle such cases.

Mutliplying polynomials. We mentioned in Section 0.2, just before Example 3, that we didn't have a very good notation for expressing the product of two polynomials. Now with double summation notation, we do. As in Section 0.2, let the two polynomials be

$$P_n(x) = \sum_{i=0}^{n} a_i x^i \qquad \text{and} \qquad Q_m(x) = \sum_{i=0}^{m} b_i x^i.$$

Our aim is to express the product of these polynomials in a simple, compact form. Here it is:

$$P_n(x)Q_m(x) = \sum_{i=0}^{m+n} \Bigl(\sum_{j=0}^{i} a_j b_{i-j} \Bigr) x^i, \qquad \begin{cases} a_j = 0, & j > n \\ b_j = 0, & j > m. \end{cases} \tag{15}$$

For example,

$$(a_1 x + a_0)(b_2 x^2 + b_1 x + b_0) = a_1 b_2 x^3 + (a_1 b_1 + a_0 b_2)x^2 + (a_1 b_0 + a_0 b_1)x + a_0 b_0. \tag{16}$$

Note that each term in parentheses on the right-hand side of (16) corresponds to one term of the summation in parentheses in Eq. (15). For each i, the inner sum in Eq. (15) consists of a sum of products of two coefficients of powers, one from $P_n(x)$ and one from $Q_m(x)$, where the sum of the powers is i. This sum of products is then the coefficient of x^i.

Product Notation

We have much less to say about product notation than summation notation for two reasons:

- Everything we've said about summation notation carries over pretty directly to product notation.
- Product notation isn't nearly as common as summation notation.

Suppose we wish to compute the product of many terms, as in

$$a_1 a_2 \cdots a_n.$$

The direct analogy with summation notation is to write this as

$$\prod_{i=1}^{n} a_i,$$

where we use $\prod$ because Pi in Greek corresponds to "P" (for product) in the Roman alphabet. Thus, for example,

$$\begin{aligned}\prod_{k=1}^{n} k &= 1 \times 2 \times 3 \cdots (n-1)n = n!,\\ \prod_{i=2}^{7} \frac{(i-1)}{i} &= \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\cdots\left(\frac{6}{7}\right) = \frac{1}{7},\\ \prod_{n=1}^{4} 2^n &= 2^1 2^2 2^3 2^4 = 2^{10}.\end{aligned}$$

If the product is empty, that is, if the predicate which defines the values of the index is true for no value of the index, then by convention, the product is 1. Why do you suppose we use this convention for products when the corresponding convention for sums is to replace an empty sum by 0?

In [17–18] we consider how to extend the idea of summation and product notation to other operators, such as set union and intersection.

Problems: Section 0.4

1. $\langle 1 \rangle$ Write the following using $\sum$ and $\prod$ notation.

a) $1 + 2 + 3 + \cdots + 100$

b) $1 \cdot 2 \cdot 3 \cdots 100$

c) $2 + 4 + 6 + 8 + \cdots + 100$

d) $1 \cdot 3 \cdot 5 \cdot 7 \cdots 99$

2. $\langle 2 \rangle$ Write the following polynomials using $\sum$ notation.

a) $x + 2x^2 + 3x^3 + \cdots + 10x^{10}$

b) $1 - x + x^2 - x^3 + \cdots + x^{10}$

c) $x + x^2 + x^3 + \cdots + x^{14}$

d) $1 + 2x^2 + 3x^4 + 4x^6 + \cdots + 8x^{14}$

3. $\langle 1 \rangle$ Evaluate the following sums.

a) $\sum_{i=2}^{4} (2i)^{-i}$

b) $\sum_{i=3}^{5} (3i + 2p)$

4. $\langle 2 \rangle$ Evaluate the following sums.

a) $\displaystyle\sum_{\substack{i=3\\ i \text{ even}}}^{8} 1/i^2$ **b)** $\displaystyle\sum_{\substack{i=3\\ 3|i \text{ or } 5|i}}^{20} (2i^2 + 6)$

5. $\langle 2 \rangle$ Evaluate each of the following sums for the values of n indicated. If the answer is particularly simple, see if you can explain why.

a) $\sum_{j=1}^{n} \left(\frac{1}{j} - \frac{1}{j+1}\right)$ $n = 4, 5.$

b) $\sum_{S \subset [n]} (-1)^{|S|}$ $n = 2, 3$ and $[n] = \{1, 2, \ldots, n\}.$

c) $\prod_{j=1}^{n} \frac{j}{j+1}$ $n = 5, 6.$

d) $\prod_{j=2}^{n} \frac{j^2-1}{j^2}$ $n = 5, 6.$

e) $\sum_{d|n} \left(\frac{d-n}{d}\right)$ $n = 6, 30.$

f) $\sum_{S \subset [n]} 2^{|S|}$ $n = 2, 3$ and $[n] = \{1, 2, \ldots, n\}.$

6. $\langle 1 \rangle$

a) Express the following inequality in summation notation:

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2) \ge (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2.$$

This is called the Cauchy-Schwarz Inequality, and it is always true.

b) Express the following inequality using sum and product notation.

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 a_2 \cdots a_n)^{1/n}.$$

This is called the arithmetic-geometric mean inequality and is true for all nonnegative a_i's.

7. $\langle 2 \rangle$ Explain why $\sum_{j=1}^{n} j2^j = \sum_{k=0}^{n} k2^k$.

8. $\langle 2 \rangle$ Rewrite $\sum_{k=1}^{10} 2^k$ so that k goes from 0 to 9 instead. Remember, a rewrite must have the same value.

9. $\langle 2 \rangle$ Rewrite $\sum_{i=0}^{n}(2i+1)$, putting it in the form $\sum_{i=?}^{?}(2i-1)$. You have to figure out what the question marks should be.

10. $\langle 2 \rangle$ Rewrite $\sum_{k=0}^{n} 3k + \sum_{j=1}^{n+1} 4j$ so that it involves just one $\sum$ sign. There may be some extra terms outside the $\sum$ sign.

11. $\langle 2 \rangle$ Rewrite $\sum_{\substack{i=1 \\ i \text{ odd}}}^{9} i^2$ in the form $\sum_{j=a}^{b} f(j)$, where you have to figure out the constants a and b and the function f.

12. $\langle 2 \rangle$ Evaluate the sums:

a) $\sum_{i=1}^{3} \sum_{\substack{j=1 \\ ij \text{ even}}}^{3} \lfloor (i/j) + (j/i) \rfloor$.

b) $\sum_{i=1}^{9} \sum_{j=1}^{\lfloor 9/i \rfloor} ij^2$.

13. $\langle 2 \rangle$ Evaluate:

a) $\sum_{j=1}^{n} \sum_{i=j}^{n} \frac{1}{i}$ $\qquad n = 3, 4.$

b) $\sum_{k=1}^{n} \sum_{j=1}^{n} \frac{(-1)^k}{j}$ $\qquad n = 3, 4.$

c) $\sum_{j=1}^{n} \sum_{i=1}^{n} \cos\left(\frac{\pi i}{2j}\right)$ $\qquad n = 3.$

14. $\langle 3 \rangle$ If $\sum_{i=0}^{100} x^i$ is squared and rewritten as $\sum_{i=0}^{n} a_i x^i$, what is a_{50}? What is n?

15. $\langle 3 \rangle$ If $\left(\sum_{i=0}^{100} x^i\right)\left(\sum_{i=0}^{25} x^i\right)$ is multiplied out, what is a_{50}?

16. $\langle 2 \rangle$ Mimic the computations in the special case displayed after Eq. (13) to prove the correctness of Eq. (13) in general.

17. $\langle 3 \rangle$ For this problem we use the following **interval notation**:

$$(a, b) = \{x \mid a < x < b\}$$

and

$$[a, b] = \{x \mid a \leq x \leq b\}.$$

Also, $\bigcup_{i=1}^{n} S_i$ means $S_1 \cup S_2 \cup \cdots S_n$ and $\bigcap_{i=1}^{n} S_i$ means $S_1 \cap S_2 \cap \cdots \cap S_n$. Evaluate the following expressions.

a) $\bigcup_{i=1}^{10} (0, i)$ **b)** $\bigcup_{j=1}^{5} [2j-2, 2j]$

c) $\bigcup_{i=1}^{\infty} (-i, i)$ **d)** $\bigcup_{k=1}^{10} [0, 1/k]$

e) $\bigcap_{k=1}^{10} [0, 1/k]$ **f)** $\bigcap_{k=1}^{10} (0, 1/k)$

g) $\bigcap_{i=1}^{\infty} [0, 1/i]$ **h)** $\bigcap_{i=1}^{\infty} (0, 1/i)$

18. $\langle 2 \rangle$ Let P_k be the statement that the integer k is a prime. Let $\vee$ stand for "or" and let

$$\bigvee_{i=1}^{n} P_i$$

mean $P_1 \vee P_2 \vee \cdots \vee P_n$. Similarly let "$\wedge$" stand for "and" and let

$$\bigwedge_{i=1}^{n} P_i$$

mean $P_1 \wedge P_2 \wedge \cdots \wedge P_n$. What do the following expressions assert and are they true?

a) $\bigvee_{k=102}^{106} P_k$ **b)** $\bigwedge_{k=1}^{3} P_{2k+1}$

c) $\bigwedge_{k=1}^{4} P_{2k+1}$

0.5 Matrix Algebra

In this section we introduce you briefly to matrices and some related concepts and notation. Matrices are very useful computational tools that organize a lot of information compactly.