SUMMATION OF SEQUENCES OF NUMBERS

Often one is interested in finding the sum of a sequences of numbers. Sometimes these follow a distinct functional form which can be represented by a single formula. One such sequence is –

$$S[n]=\{1,3,6,10,15,21,28,\ldots\}$$

To generate a functional form we first write down an array of numbers representing various differences. This array reads-

```
1   3   6  10   15   21   28
  2   3   4   5    6    7        1st diff.
    1   1   1   1    1           2nd diff.
      0   0   0   0              3rd diff.
```

This suggests we have a form $S[n]=A+Bn+Cn^2$ with no cubic term possible. It means we must have –

$$1=A+B+C$$
$$3=A+2B+4C$$
$$6=A+3B+9C$$

On solving, this produces the formula-

$$S[(n]=n(1+n)/2$$

From this result we see that S[n)] is just the sum of the first n integers. That is, S(5)=15 and S(7)=28. The first hundred integers sum to S(100)=5050. It is our purpose in this article to find general formulas for difference equations for a variety of other infinite sequences.

Let us begin with the sequence-

$$S[n]=\{1,5,14,30,55,91,140,\ldots\}$$

Here the corresponding difference array reads-

```
1    5    14    30    55    91    140
   4    9    16    25    36    49
      5    7     9     11    13
         2     2     2     2
```

This form suggests we have a S(n) formula which is a cubic-

$$S(n)=A+Bn+Cn^2+Dn^3$$

The corresponding equations read-

$1=A+B+C+D$, $5=A+2B+4C+8D$, $14=A+3B+9C+27D$, and $30=A+4B+16C+64D$

On solving we have A=0, B =1/6, C=1/2, and D=1/3. It produces the formula-

$$S(n)=(n+3n^2+2n^3)/6$$

One sees that S(2)=5, S(10)=2310, and S(100)=230100. That is, this series represents the sum of the squares of the first n integers.

From the two sequences given above, we can conclude that a formula representing the sum of the first n integers taken to the pth power each produces a polynomial formula in powers through p+1.

Often one encounters formulas for S[n] which contain more than just a polynomial in n. Let us demonstrate the sequence governed by the formula-

$$S[n]=(1+n)2^{n-2}$$

Writing out the sequence, we find-

S[n]={1,3, 8,20, 48,112, …}

Here a difference array of S[n] shows no regularity meaning we are dealing with a function other than just a polynomial. So we look at any other regularities one can find. We note that –

S[n+1]=2S[n]+2^(n-1) with S[1]=1

This represents a difference equation whose solution yields S[n]. Using the one line computer program-

```
S[1]:=1;  for n from 1 to 10 do S[n+1]:=evalf(2*S[n]+2^(n-1))od;
```

We find-

S[1] := 1, S[2] := 3, S[3]:= 8, S[4]=20, S[5] := 48, S[6] := 112, S[7] := 256.

S[8] := 576, S[9]:=1280, S[10]=2816

These each satisfy the formula S[n]=(1+n)2^(n-2). Here S[100]=101*2^98.

Several other sequences are recoverable from the modified Pascal Triangle which we discovered about five years ago. Here is its form-

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 4 & & 1 & & & & \\
 & & & 1 & & 11 & & 11 & & 1 & & & \\
 & & 1 & & 26 & & 66 & & 26 & & 1 & & \\
 & 1 & & 57 & & 302 & & 302 & & 57 & & 1 & \\
1 & & 120 & & 1191 & & 2416 & & 1191 & & 120 & & 1
\end{array}
$$

$$\boxed{D[n,m]=\sum_{k=1}^{n}\frac{(-1)^{k-1}(m+1)!(n+1-k)^{m}}{(k-1)!(m+2-k)!}}$$

Note that the sum of the elements in each row m equal exactly m!

Let us first consider the sequence given by-

S[n]={1, 4, 11, 26, 57, 120,…}

We note that S[3]=11=2S[2]+3 and S[4]=26=2S[3]+4. It allows us to write the difference equation-

S[n+1]=2S[n]+n+1   subject to S[1]=1

Running our computer search program yields the first ten values of S[n] given by-

S[n]= {1, 4, 11, 26, 57, 120, 247, 502, 1013, 2036,….}

It is also possible to run things backwards starting with say –

S[n]=n2^(n-1)

This produces the sequence-

S[n]={1,4,12,32,80,192,448,1024,5120,..}

From it we also have the difference equation-

S[n+1]=2*S[n]+2^n   subject to S[1]=1

The closed form solution to this equation is S[n]=n2^(n-1).

As a final S[n] formula consider-

S[n+1]=-1+n^2+2^(n-1) subject to S[1]=1

Writing out the first ten elements we get-

S[n]={1, 5, 12, 23, 40, 67, 112, 191, 336, 611, …}

From this we also have the difference equation-

$$S[n+1-S[n]=1+2n+2^{(n-1)} \quad \text{subject to} \quad S[1]=1$$

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