Teaching the concept of function: Definition and problem solving

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The present study investigates students' abilities to understand the concept of function. Secondary education students were asked to (i) define the concept of function and present examples of functions, (ii) translate between different representations of a function and (iii) solve function problems. Findings revealed students' great difficulties in proposing a definition of function, in solving tasks of conversions between different modes of representation, and in solving function problems. Based on the students' abilities and misconceptions about functions, teaching practices for improving the students' understanding of functions are discussed.

Keywords: Function, definition, use of representations, teaching methodology.

INTRODUCTION

For more than twenty years, the concept of function has been internationally considered as a unifying theme in mathematics curricula (Steele, Hillen, & Smith, 2013). Students face many instructional obstacles when developing an understanding of functions (Sajka, 2003; Sierpinska, 1992). Kieran (1992) questions whether students' inability of conceptually understanding functions is related to its teaching or is due to students' inappropriate way of approaching function tasks. Sajka (2003) indicates that students' abilities in solving tasks involving functions are influenced by the typical nature of school tasks, leading to the use of standard procedures. According to a standard didactic sequence, students are asked to infer the properties of a function using the given graph, by following a specific procedure.

In relation to the above, our study examines students' conceptions of function, as it is one of the most important topics of the curriculum and it is related to other subjects, such as physics (Sanchez & Llinares, 2003). In fact, the results we present in this paper are a part of a large scale cross-sectional study examining the use of different modes of representations in functions at the secondary school level. Adopting a developmental perspective at different grades of secondary education, we aim to trace students' abilities in defining functions, recognizing and manipulating them across representations and in problem solving, emphasizing the approach used (algebraic or geometric). Thus, our main questions are: (i) What abilities do students have to define and flexibly manipulate functions and solve function problems?, and (ii) What are the differences in students' performance at the 3rd, 4th and 5th grades of secondary education? Based on our results, we provide suggestions for teaching practices that can facilitate students' understanding of functions.

THEORETICAL FRAMEWORK

The role of multiple representations in the understanding of functions

There is strong support in the mathematics education community that students can grasp the meaning of a mathematical concept by experiencing multiple mathematical representations of that concept (Sierpinska, 1992). One of the main characteristics of the concept of functions is that they can be represented in a variety of ways (tables, graphs, symbolic equations, verbally) and an important aspect of its understanding is the ability to use those multiple representations and translate the necessary features from one form to another (Lin & Cooney, 2011). In order to be able to use the different forms of representations as tools in order to construct a proof, students have to understand the basic features of each representation and the limitations of using each form of representation.

According to Steele and colleagues (2013), it is typical in the U.S. for the definition of a function and a connection to the graphic mode of the function concept to occur in the late middle grades, while the formal study of function with an emphasis on symbolic and graphical forms occurs in high school. According to Bardini, Pierce and Stacey (2004), in their brief overview of Australian school mathematics textbooks, symbolic equation solving follows graphical work. A proper understanding of algebra, however, requires that students be comfortable with both of these aspects of functions (Schwartz & Yerushalmy, 1992). It is thus evident that the influence of teaching is extremely strong, and the promotion of specific tools or processes enforces the development of specific cognitive processes and structures. For example, it has been suggested that one way to improve learners' understanding of some mathematical concepts might be the use of graphic and symbolic technologies (Yerushalmy, 1991).

The role of definition in the understanding of functions

In mathematics definitions have a predominant role in the construction of mathematical thinking and conceptions. Vinner and Dreyfus (1989) focus on the influence of concept images over concept definitions. Over time, the images coordinate even more with an accepted concept definition, which in turn, enhances intuition to strengthen reasoning. A balance between definitions and images, however, is not achieved by all students (Thompson, 1994). Actually, pupils' definitions of function can be seen as an indication of their understanding of the notion and as valuable evidence of their mistakes and misconceptions. Elia, Panaoura, Eracleous and Gagatsis (2007) examined secondary pupils' conceptions of function based on three indicators: (1) pupils' ideas of what function is, (2) their ability to recognize functions in different forms of representations, and (3) problem solving that involved the conversion of a function from one representation to another. Findings revealed pupils' difficulties in giving a proper definition for the concept of function. Even those pupils who could give a correct definition of function were not necessarily able to successfully solve function problems.

METHODOLOGY

Participants

The participants were 756 secondary school students from eight schools in Cyprus. There were 315 students at 3rd grade of gymnasium (15 years old), 258 students at 4th grade of lyceum (16 years old) and 183 students at 5th grade of lyceum (17 years old). The students at 5th grade follow a scientific orientation and attend more advanced levels of mathematics courses. Students' participation was due to the voluntarily participation of their teachers, thus our sampling procedure was not randomized.

Procedure

Two tests including different types of tasks were developed. The tests were developed by the researchers and secondary mathematics teachers. The tasks were mainly aligned to the level of the 3rd graders and the tests were piloted to almost 100 students at each grade. At the middle of the school year, each test was administered to students by a researcher or by a teacher who had been instructed on how to correctly administer the test. The testing period was 40 minutes. The tests were scored by the researchers, using 1 and 0 for correct and wrong answers, respectively.

The tasks focused on (a) defining and explaining the concept of function, (b) recognizing, manipulating and translating functions from one representation to another (algebraic, verbal and graphical), (c) solving problems.

Costas has €20 and spends €1 per day. His sister has €15 and spends €0,5 per day.

i. Find the function expressing the amount of money (y) each person will have in relation to the number of days (x).
ii. Design the graph showing this function for each person.
iii. In how many days will the two brothers have the same amount of money? What will this amount of money be?

Figure 1: Example of a problem solving task

The first test asked students to (i) present a definition of function and an example, (ii) explain their procedure for recognizing that a graph does not represent a function and present a non-example of function, (iii) write the symbolic representation of six verbal expressions (e.g. "the area E of a square in relation to its side"), (iv) draw a graph to solve a problem, (v) recognize graphs of functions, (vi) explain a graph in terms of the context, and (vii) examine whether symbolic expressions and graphs represent functions. In the second test students had to (i) present their procedure for examining whether a graph represents a

function, (ii) draw graphs of given functions, (iii) recognize graphs of given verbal or symbolic expressions, and (iv) write the symbolic equations of given graphs. The reliability for the total of the items on both tests was high (Cronbach's alpha=0.868).

RESULTS

According to our research questions, we first present the results of students' performance on defining the concept of function and their ability to present an example in order to explain the definition. Then, we concentrate our attention on the procedure students follow in order to justify whether a verbal expression or an equation is a function. A crosstabs analysis gives further insight into the performance of students in defining and recognizing functions. Finally, differences between students in the three grade levels are presented.

Based on the results of our descriptive analysis, 159 students were able to present a correct definition, 242 presented a wrong definition, and 358 students did not give any answer. Most of the students who presented a correct definition (Table 1) were at the 4th and 5th grade. The results were similar to the task that asked students to present an example for explaining the definition of function. Only 401 of the students presented an example, and of these, 323 were correct. The highest percentages of correct examples (Table 1) were at the 4th and 5th grade. This result is the first indication that students are perhaps more capable of providing an example in order to explain a mathematical concept than of defining it verbally by using formal language and/or symbolization.

In the first test students were asked to explain their procedure for identifying a graph that does not represent a function and provide a relation that does not represent function. In the second test, students were asked to explain their procedure for identifying a graph that represents a function and to provide an example of a function. Table 2 indicates the percentages of correct answers for the specific tasks. In all cases, the results were higher in the 5th grade, as was expected, but there were especially negative results for students in the 3rd grade. The majority of the 3rd graders did not give an answer and many of those who answered presented a wrong procedure.

	Correct definition	Correct example
Grade 3 (N=315)	8.2%	23.0%
Grade 4 (N=258)	58.5%	57.0%
Grade 5 (N=183)	32.7%	55.7%

Table 1: Students' correct definitions and examples for each grade

Tasks	3rd (%)	4th (%)	5th (%)	Total (%)
Procedure indicating that a verbal expression or an equation is not function	2.2	38.4	42.0	24.3
Procedure indicating that an expression is function	2.5	32.6	47.5	23.7
An example of a function	19.4	50.4	55.7	38.6
A non-example of a function	3.8	38.4	38.3	23.9

Table 2: Percentages of students' correct answers at specific tasks

We analysed, by using crosstabs analysis (Table 3), the performance of students who correctly described a procedure for determining whether a graph represents a function in relation to their ability to correctly identify functions presented graphically. Results indicated that less than half of the students, who correctly described a procedure, correctly recognized the graph that represented a function (44.1%).

We further analysed, by using crosstabs analysis (Table 3), the characteristics of students who were able to correctly define function. Consequently, these students seemed to have a more conceptual understanding of function. Results indicated that 89.4% of the students presented both a correct definition of function and a correct example, showing that they likely have a strong theoretical understanding of the concept. At the same time, of the students that provided a correct definition of function, 74.2% also succeeded in describing a procedure for determining whether a graph did not represent a function, and 62.7% also succeeded in describing the procedure for recognizing whether a graph did represent a function. These initial results permit us to assume that the students who are able to define function and explain by providing an example are the students with the highest conceptual understanding.

In the second test, students who were able to correctly describe a procedure for determining whether a graph represented a function were generally able to present an example of function (90.4%). Of the students who were able to correctly describe a procedure for determining whether a graph did not represent a function, 83.7% were also able to present a symbolic non-example of a function. It is important to note, however, that of the total sample, only 153 students correctly presented a procedure for recognizing a graph that represented a function, while only 135 students correctly presented a procedure for recognizing a graph that did not represent a function. Similarly, only 241 students correctly presented an example of a function, and only 165 students correctly presented a non-example.

	Example for definition	Describing a procedure for determining whether a graph represents a function	Describing a procedure for determining whether a graph does not represent a function
Recognition of function	—	44.1%	—
Correct definition	89.4%	62.7%	74.2%
Function Example	—	90.4%	—
Function Non-example	—	—	83.7%

Table 3: Crosstabs analysis for students' (total sample) understanding of functions

The second objective of the study was to identify statistically significant differences ($p<0.05$) concerning the concept of function in respect to students' grade level in order to investigate the developmental aspect of a conceptual understanding of function. ANOVA analysis was used for comparing students' means in presenting the definition of the concept and giving an example of a function in respect to the categorical variable of their grade. In the first case, Scheffé analysis indicated that there was a statistically significant difference ($F_{2,397} = 33.396$, $p<0.01$) between the students at the 3rd grade with the students at the 4th and 5th grade ($\bar{x}_3 = 0.11$, $\bar{x}_4 = 0.53$, $\bar{x}_5 = 0.48$). Although the highest mean was at the 4th grade, there was not any statistically significant difference between the two grades at the lyceum. Concerning the presentation of an example in order to explain the definition of the concept, the difference was statistically significant ($F_{2,398} = 5.896$, $p<0.01$) only between the 3rd and the 4th grade ($\bar{x}_3 = 0.72$, $\bar{x}_4 = 0.88$, $\bar{x}_5 = 0.78$). The same analysis was conducted concerning students' ability to describe a procedure for recognizing an equation which did not represent a function, in respect to grade level. There were statistically significant differences between the 3rd grade and the two other grades ($F_{2,292} = 33.510$, $p<0.01$) with a large difference between the means ($\bar{x}_3 = 0.12$, $\bar{x}_4 = 0.47$, $\bar{x}_5 = 0.55$).

Items from both tests were grouped in order to be able to further analyse students' performance concerning the specific aspects which were the main interest of this study: (i) Propose definition, (ii) present examples to explain a concept or a procedure, (iii) recognition of the concept, (iv) translation of the concept from one representation to another, (v) construction of graphs which represent functions, as an indication of manipulating the concept and (vi) problem solving tasks. Table 4 presents the means and standard deviations of students' performance on these specific dimensions.

	$\bar{X}$	SD
Definition	0.718	0.261
Examples	0.837	0.231
Recognition	0.449	0.152
Translation	0.619	0.212
Construction	0.648	0.354
Problem solving	0.393	0.169

Table 4: Means and standard deviations of students' performance

ANOVA analysis was conducted for examining the statistically significant differences for each of the above aspects of a conceptual understanding of function in respect to the three grade levels. Statistically significant differences ($p<0.05$) were found only concerning three of the dimensions. There was a statistically significant difference ($F_{2,182} = 7.678$, $p<0.01$) concerning the proposed definition of the concept. The Scheffé analysis indicated that the difference was between the 3rd grade with the 4th and the 5th grade ($\bar{x}_3 = 0.33$, $\bar{x}_4 = 0.75$, $\bar{x}_5 = 0.71$). The second statistically significant difference was for the students' ability to recognize functions presented in different forms of representations ($F_{2,570} = 18.926$, $p<0.01$). The difference was between the students at the 3rd grade in comparison to the students at the 4th and 5th grade ($\bar{x}_3 = 0.48$, $\bar{x}_4 = 0.41$, $\bar{x}_5 = 0.40$). It

was unexpected that in this case the performance at the 3rd grade was higher than the two other grades. The third statistically significant difference was for the students' ability to translate the functions from one type of representation to another ($F_{2,165} = 11.077$, $p<0.01$). The students at the 3rd grade had a lower performance than the older students ($\bar{x}_3 = 0.343$, $\bar{x}_4 = 0.547$, $\bar{x}_5 = 0.702$).

DISCUSSION

We have concentrated on students' ability to define the concept of function and on their ability to handle flexibly the different modes of representation of function. The results of the present study confirm previous findings that students face many difficulties in understanding function at different ages of secondary education (Sajka, 2003; Tall, 1991). Findings revealed serious student difficulties in proposing a proper definition for function or a tendency to avoid proposing a definition due to their possible belief that the intuitive and informal presentations of their conceptions cannot be part of the mathematical learning. Those difficulties are consistent with Elia and colleagues' (2007) findings. Although formal definitions of mathematical concepts are included in the mathematics textbooks for secondary education, mathematics teachers do not focus on definitions. Instead they promote the use of algorithmic procedures for solving tasks, as actions and processes (Cottrill et al., 1996), and underestimate the word and meaning (Morgan, 2013) as interrelated dimensions of the concept.

Secondly, it seems easier for students to present an example for explaining a mathematical concept, rather than using a non-example in order to explain the respective negative statement. At the same time, it is easier for them to use an example for explaining the concept of function rather than explaining the procedure they follow for determining whether a graph represents a function. This is probably a consequence of the teachers' method of using examples in order to explain an abstract mathematical concept. Thirdly, students had a higher performance on manipulating the concept in graphical form and translating from one type of representation to another, than on recognizing functions in algebraic and graphical forms. Symbolic equation solving follows the graphical work (Llinares, 2000), and probably for this reason, the performance on graphical functions was higher. Finally, the results of the students at the 3rd grade of secondary education were especially negative, thus we have to further examine whether their processing efficiency and cognitive maturity prevent us from teaching the specific concept at that age. Despite the tendency to use the spiral development of concept in the teaching process and the curriculum (Ministry of Education and Culture, 2010), we have to rethink the teaching methods we use at the different ages and the cognitive demands of tasks at each age.

We believe that it is not adequate just to describe the students' knowledge of a concept, but it is interesting to design and implement didactic activities and examine their effectiveness. Brown (2009) suggests that the construction of concept maps will enable teachers to have in mind all the necessary dimensions of the understanding of the concept. The distinction of the procedural understanding and the conceptual understanding and the lack of interrelations between aspects such as the definition of the concept, the manipulation of the concept and problem solving related to a concept have as a possible result the phenomenon of compartmentalization (Elia, Gagatsis, & Gras, 2005). Thus, the use of multiple representations, the connection, coordination and comparison with each other and the relation with the definition of the concept should not be left to chance, but should be taught and learned systematically.

The verification of the interrelations between the different aspects of understanding functions are within our next steps, which include the confirmatory factor analysis (CFA), for confirming the theory about the structural organization of the conceptual understanding of functions. We are going to examine whether the students' abilities in defining the concept, in recognizing and manipulating the concept and in translating the concept from one representation to another are important dimensions of the conceptual understanding of functions. Specifically we will focus on the interrelationship between these dimensions. Special emphasis will be given to the role and influence of the definition on the remaining dimensions of the conceptual understanding of functions, as our results revealed students' great difficulties in this particular aspect of the concept.

Concluding, the present study enables us to know and understand how students conceptualize the notion of function and to realize the students' obstacles and misunderstandings. Teaching processes and teaching materials need to be enriched with problem solving

situations. The given examples presented by students were mathematically structured by using a formalistic way and there was not any reference to a daily experience. In the context of the interdisciplinary social reality, the concept of function has to be related to other relevant domains such as physics, engineering, and technology.

Limitations

A limitation of our study concerns the administration of the tests. In cases where the administration was not performed by the researchers, there is no certainty that the proper amount of time was given to the students. Thus we cannot be sure that the same conditions held in every classroom during the administration of the tests, and this may have affected the reliability of the tests. The teachers who administered the tests, however, were provided with all the necessary instructions. A further limitation of the study was our inability to control the teaching method which was used for the specific concept. In Cyprus, however, teachers receive the same instructions by the Ministry of Education for the teaching methods they have to use in their classes and the same in-service training. There is also a common curriculum and a common textbook for students.

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