

PRECALCULUS

THE LANGUAGE OF FUNCTIONS*

In algebra you began a process of learning a basic vocabulary, grammar, and syntax forming the foundation of the language of mathematics. For example, consider the following algebraic sentence:

$$\text{Solve for } x \text{ if } x^2 - 3x + 5 = 8$$

You learned to read this sentence, understand what it means, what it is asking for, and in addition, how to go about answering the question.

In precalculus, it is important to learn and understand the language of functions. This includes understanding function notation and sentences using function notation. Using the language of functions, for example, we can restate the same question above in the following way:

If $f(x) = x^2 - 3x + 5$ when does $f(x) = 8$? **OR**

If $f(x) = x^2 - 3x + 5$ for what value(s) of x does $f(x) = 8$? **OR**

If $f(x) = x^2 - 3x + 5$ and $g(x) = 8$ for what value(s) of x does $f(x) = g(x)$? **OR**

If $f(x) = x^2 - 3x + 5$ and $g(x) = 8$ when does $f(x) = g(x)$?

Using the language of functions, these are just a few ways to express the same question first stated above “algebraically”. What is important to note here is that the methods for finding the answer to the question may be the same as in algebra (and the process of finding the answer may be easy once you understand what is being asked): the difficult part may be understanding what is being asked.

That is why it is important for you to *understand* functions and function notation.

I. You should understand what makes an assignment a function. You should be able to identify if an assignment is a function whether the function is given as an assignment using sets, tables, ordered pairs, equations, or graphs. You should be able to identify domains (including natural domains) and ranges using sets, tables, ordered pairs, equations, or graphs. You should also know how to (and when to) use the graphing calculator to help answer questions about functions.

*This handout is not intended as a tutorial or a complete review of functions, but rather as a quick review of notation that we use frequently in this course. There are additional important topics such as applications, quadratic functions, extreme values, etc., which are discussed in lecture and in the text, but are not covered here.

EXAMPLES: The following are different ways functions can be defined. Make sure you are familiar with each type of representation, and that you understand *why* these assignments are functions

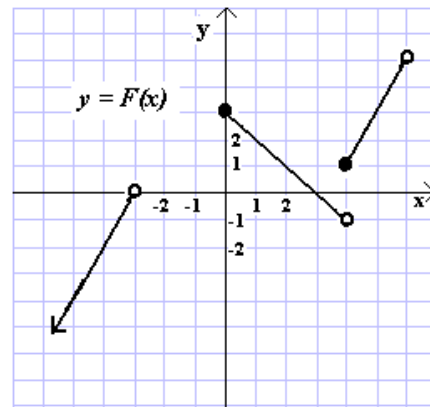
- A. $f(x)$ defined by $\begin{matrix} x & f(x) \end{matrix}$ or equivalently, f , defined by the set $f = \{ (2, 3.1), (4.2, -2), (5.8, 7), (6.8, 3.1) \}$

2	3.1
4.2	-2
5.8	7
6.8	3.1

B. $h(x) = x^2 - 2x + 3$

C. $H(x) = \frac{1}{\sqrt{3 + 5x - 2x^2}}$

- D. $F(x)$ defined by the graph on the right:

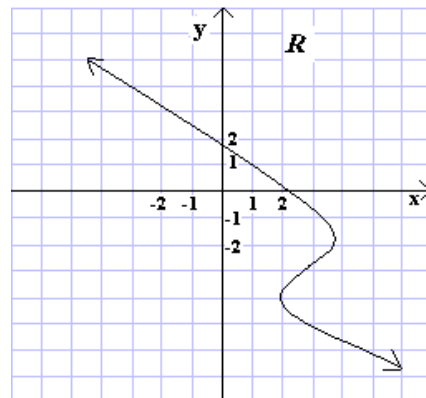


The following are NOT functions (make sure you understand why):

E. $y^2 = x^2 + 2$

- F. K defined by the set of ordered pairs: $\{(2,5), (3,8), (2,4)\}$.

- G. R defined by the graph on the right:



Q1. For the examples of functions given above, find their domain and range.

A1. For $f(x)$: Domain = $\{2, 4.2, 5.8, 6.8\}$; Range = $\{-2, 3.1, 7\}$.

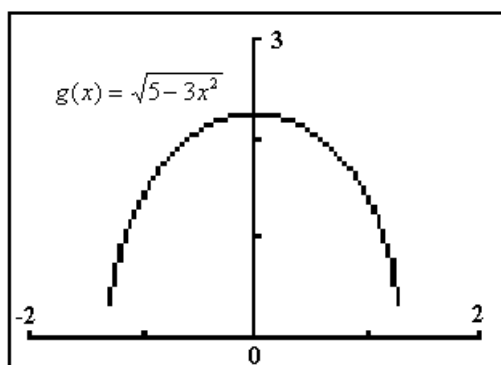
For $h(x)$: Domain = $(-\infty, \infty)$; Range = $[2, \infty)$ [To find the range in this case, you will need to know the graph of a quadratic function or use a graphing calculator.]

For $H(x)$: Domain = $(-1/2, 3)$; Range (accurate to 3 places) = $(0.404, \infty)$ [You should be able to find the domain by algebraic methods AND by using a graphing calculator. To find the range in this case, you will need to use a graphing calculator and apply your knowledge of functions to interpret the graph on the calculator.]

For $F(x)$: Domain = $(-\infty, -3) \cup [0, 6)$; Range = $(-\infty, 5)$ [Note that 4 is included in the domain.]

Q2. Find the domain and range of $g(x) = \sqrt{5 - 3x^2}$ accurate to one decimal place.

A2. Using a graphing calculator (and your knowledge of the function so that you can interpret the calculator graph shown below), we get Domain = $(-1.3, 1.3)$; Range = $(0, 2.2)$. [You should also be able to find the domain algebraically as well.]



II. You should understand basic function notation.

EXAMPLES: For functions given in A, B, and C above, $f(2) = 3.1$, $h(3) = 6$,

$H(1) = \frac{1}{\sqrt{6}}$, and $H(5)$ is not defined (Why?). For $F(x)$ defined above in part D,

$F(-4) = -2$, $F(-3)$ is not defined, $F(-1)$ is not defined, $F(0) = 3$, $F(2) = 1$, $F(4) = 1$ (Why?), and $F(6)$ is not defined.

Q3. If $f(x) = 2x^2 - 4x - 5$, find $f(x-3)$, $f(x) - 3$, and $f(x) - f(3)$.

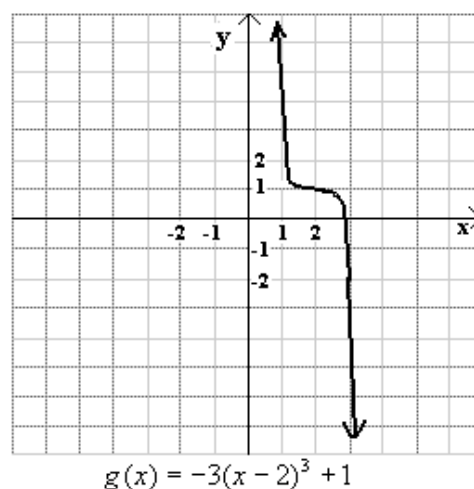
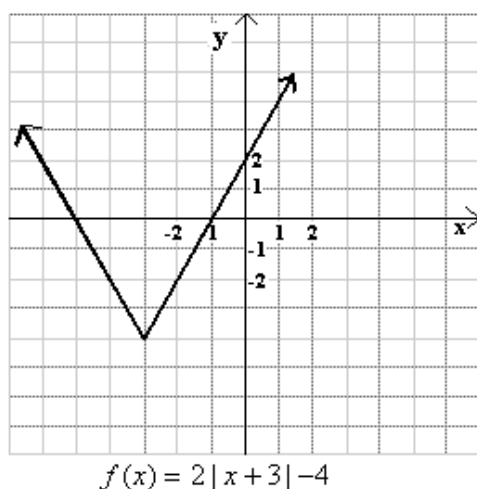
A3. $f(x-3) = 2x^2 - 16x + 25$, $f(x) - 3 = 2x^2 - 4x - 8$, $f(x) - f(3) = 2x^2 - 4x - 6$.

Q4. If $f(x) = x^{-2}$, find $\frac{f(x+h) - f(x)}{h}$

A4.
$$\frac{f(x+h) - f(x)}{h} = \frac{-2x-h}{x^2(x+h)^2}.$$

III. You should know graphs of the basic functions. (Without a graphing calculator) you should be able to sketch the graphs of functions by reflections, translations, and stretching, and be able to evaluate and graph piecewise-defined functions.

EXAMPLES: WITHOUT a graphing calculator, you should be able to sketch the graph of $f(x) = 2|x+3| - 4$ and $g(x) = -3(x-2)^3 + 1$ shown below.

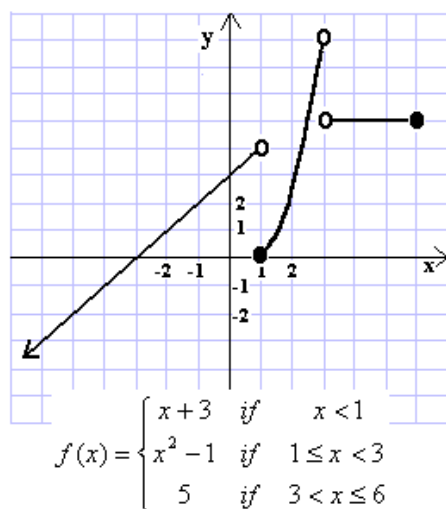


The x -intercepts (zeros) of $f(x)$ are $x = -1$ and $x = -5$; the y -intercept of $f(x)$ is $y = 2$. The zero of $g(x)$ is $x = 2 + \sqrt[3]{\frac{1}{3}}$; the y -intercept is $y = 385$.

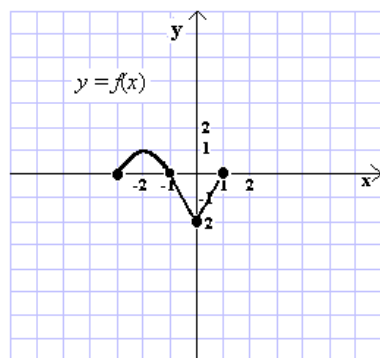
Given the piecewise-defined function $f(x)$ defined below,

$$f(x) = \begin{cases} x+3 & \text{if } x < 1 \\ x^2 - 1 & \text{if } 1 \leq x < 3 \\ 5 & \text{if } 3 < x \leq 6 \end{cases}$$

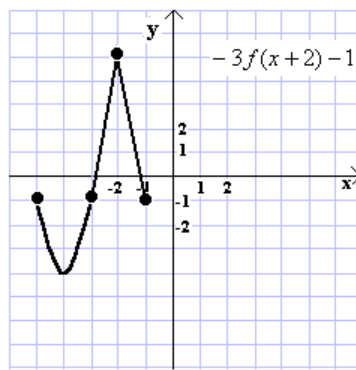
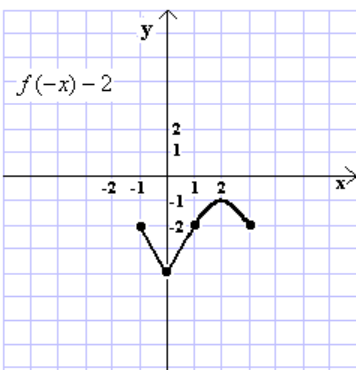
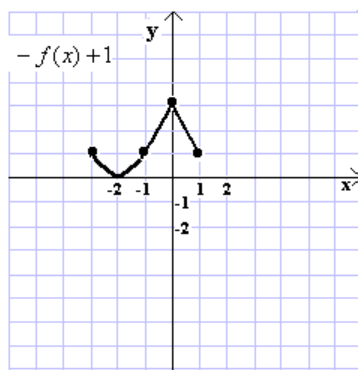
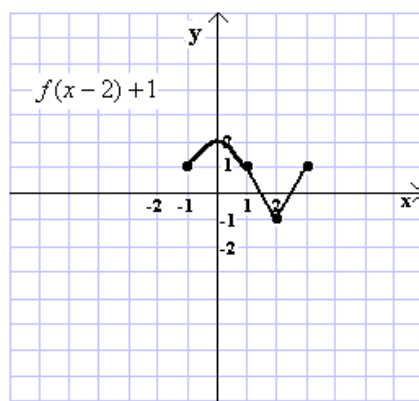
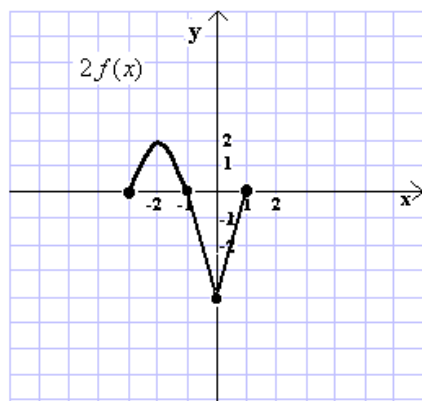
then $f(-4) = -1$, $f(0) = 3$, $f(1) = 0$, $f(2) = 3$, $f(3)$ is not defined, $f(4) = f(4.5) = f(6) = 5$, and $f(8)$ is not defined. The domain of $f(x)$ is $(-\infty, 3) \cup (3, 6]$. The graph appears on the next page.



Q5. Given the graph of $y = f(x)$ on the right, sketch the graph of $2f(x)$, $f(x-2)+1$, $-f(x)+1$, $f(-x)-2$, and $-3f(x+2)-1$



A5.



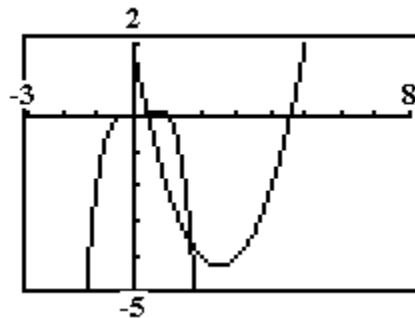
IV. You should be able to algebraically determine if a function is symmetric with respect to the y-axis, origin, or neither.

EXAMPLE: The function $f(x) = \frac{1}{x^2 + 1}$ is symmetric with respect to the y-axis (Why?):

the function $g(x) = x^3 + 2x - 1$ is neither symmetric with respect to the origin nor symmetric with respect to the y-axis (Why?).

V. You should be able to solve equations and inequalities using graphs of functions on a graphing calculator.

EXAMPLE: If $f(x) = x^2 - 5x + 2$ and $g(x) = x^3 - x^4$, one way to solve the equation $f(x) = g(x)$ or the inequality $f(x) > g(x)$, is to use a graphing calculator to graph the two functions, (shown below) and read off the solutions:



The solution to $f(x) = g(x)$ (to 3 places) is $x = 0.428$ and $x = 1.718$. The solution to $f(x) > g(x)$, (to 3 places) is $(-\infty, 0.428) \cup (1.718, \infty)$. Why?

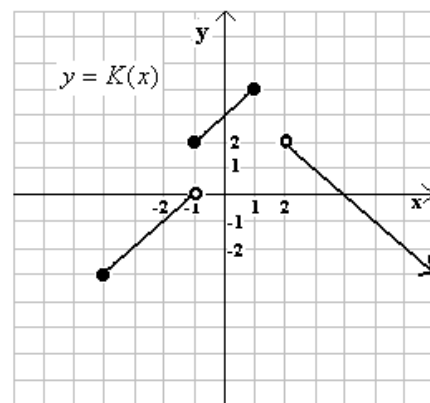
VI. You should understand what it means to combine functions by operations. You should understand the meaning of composition of functions, and how to compose functions whether the functions are defined with tables, graphs, or formulas, or other functions.

EXAMPLES: Consider the functions defined below:

$f(x)$ defined by

x	$f(x)$
1	2.8
3	4.2
4	-1
5	-3.1

$K(x)$ defined by the graph below:



$$g(x) = 2x - 5, \quad h(x) = \sqrt{x + 7}, \quad F(x) = x^2 - 5$$

Then

$(f \circ g)(3) = 2.8$, $(g \circ f)(3) = 3.4$, $f \circ g(-1)$ is not defined, $(h \circ g)(4) = \sqrt{10}$, $(g \circ h)(2) = 1$, $(h \circ F)(-1) = \sqrt{3}$, $(F \circ h)(-9)$ is not defined, $(K \circ F)(2) = 2$, and $(F \circ K)(1) = 11$.

The formula for $(h \circ F)(x) = \sqrt{x^2 + 2}$ and the formula for $(F \circ h)(x)$ is $(F \circ h)(x) = x + 2$, where $x \geq -7$ [The restricted domain for $(F \circ h)(x)$ is necessary. Why?]

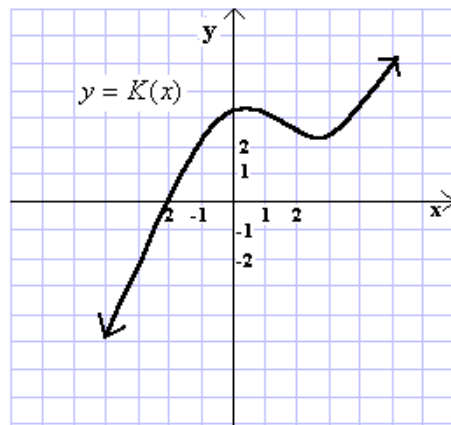
If we define the function $G(x)$ as follows:

$$G(x) = \frac{K(x)}{x^2 + 2}, \text{ then}$$

$$G(3) = \frac{1}{11} \text{ and } G(-2) = -\frac{1}{6}.$$

VII. You should understand what a 1-1 function is, and be able to determine if a function is 1-1 whether the function is defined with a table, graph, or formula.

EXAMPLES: The functions $f(x) = 3x^3 - 1$, $g(x) = \sqrt{x-2}$, and $f(x) = \frac{1}{x^3}$ are 1-1: the function $h(x) = 3x^2 - 4x + 1$ is **not** 1-1. The graph of $K(x)$ below is **not** 1-1.



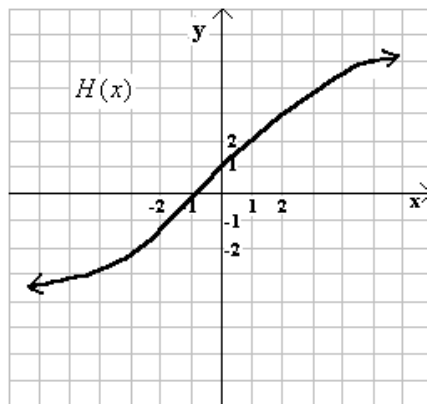
VIII. You should understand what the inverse of a function is, and be able to determine whether a function has an inverse. You should be able to find the inverse of a function whether the function is defined with a table, graph, or formula, and determine its domain and range. You should be able to evaluate the inverse $f^{-1}(a)$ for numerical values of a whether the function is defined with a table, graph, or formula (as well as by using a graphing calculator).

EXAMPLES: The following functions have inverses:

$f(x)$ defined by

x	$f(x)$
1	7
3	8
4	-5
5	-3.1

$H(x)$ defined by the graph below:

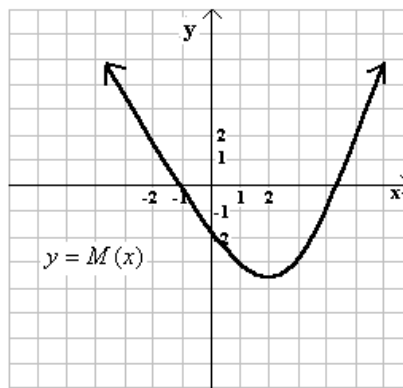


$$g(x) = 2x^3 - 4 \quad h(x) = \frac{x-2}{4x+2}$$

The following functions do **not** have inverses:

$$K(x) = 3x^2 - 5$$

$M(x)$ defined by the graph on the right:



Q6. Given the functions (with inverses) defined above, find $f^{-1}(8)$, $g^{-1}(3)$, $h^{-1}(3)$, and $H^{-1}(3)$.

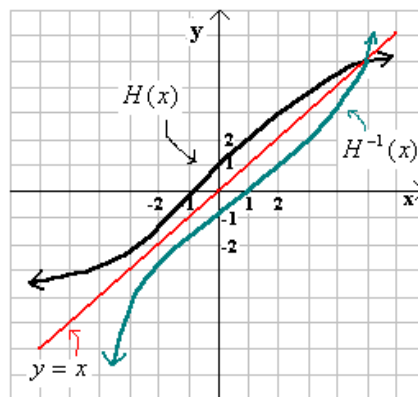
A6. $f^{-1}(8) = 3$, $g^{-1}(3) = \sqrt[3]{\frac{7}{2}}$, $h^{-1}(3) = -\frac{8}{11}$, and $H^{-1}(3) = 2$.

Q7. Find the inverses of the functions (with inverses) above.

A7. $f^{-1}(x)$ is defined by

x	$f^{-1}(x)$
-5	4
-3.1	5
7	1
8	3

$y = H^{-1}(x)$ is below:



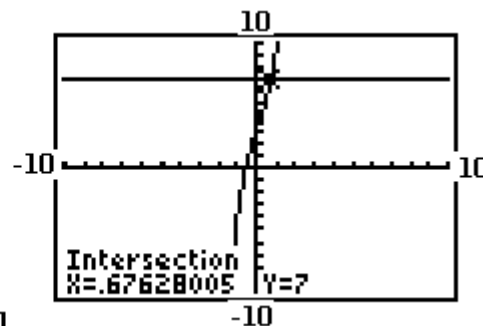
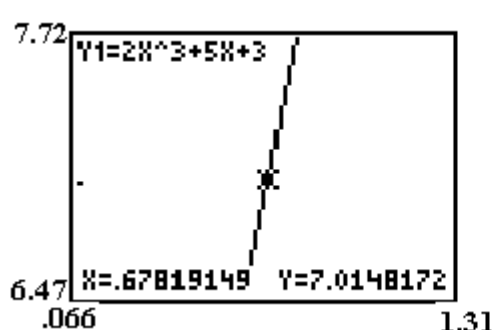
$$g^{-1}(x) = \sqrt[3]{\frac{x+4}{2}}, \quad h^{-1}(x) = \frac{2x+2}{1-4x}.$$

The domain of $f^{-1}(x)$ is $\{-5, -3.1, 7, 8\}$; the range is $\{1, 3, 4, 5\}$.

The domain and range of $g^{-1}(x)$ is $(-\infty, \infty)$. The domain of $h^{-1}(x)$ is $\left(-\infty, \frac{1}{4}\right) \cup \left(\frac{1}{4}, \infty\right)$.

Q8. Given $f(x) = 2x^3 + 5x + 3$, find $f^{-1}(7)$ rounded to two decimal places.

A8. There are a few ways to identify $f^{-1}(7)$ using a graphing calculator: one way is to zoom in on the graph of $f(x) = 2x^3 + 5x + 3$ where $y = 7$ (the figure below and left). Another way is to graph the functions $f(x) = 2x^3 + 5x + 3$ and $y = 7$ shown below on the right. Hence $f^{-1}(7) = 0.68$ rounded to two decimal places.



MORE EXAMPLES: $f(x), g(x), h(x)$, and $H(x)$ defined on page 8, then

$$(g^{-1} \circ f)(3) = \sqrt[3]{6}, \quad (h \circ g^{-1})(12) = 0, \quad \text{and} \quad h \circ H^{-1}(2) = -\frac{1}{6}. \quad (\text{Why?})$$