The Summation Notation

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Intermediate Econometrics / Forecasting Class Notes
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Definition and Rules for Sums

The summation notation "$\sum$" is used to denote summation over an index. For an arbitrary set of numbers $\{x_1, x_2, ..., x_n\}$, define

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + ... + x_n$$

The index i is called the **summation index**, and the first and last values of the index are called the **summation limits**.

Examples

(1) The simple average of a set of numbers $\{x_1, x_2, ..., x_n\}$ is written $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$.

(2) Write the sum $4 + 8 + 12 + 16 + 20 + 24$ in summation notation. Ans: $\sum_{i=1}^{6} 4i$. Another possibility is $\sum_{i=0}^{5} 4(i+1)$: Summation representations are not unique.

In the first example above, the summation index i is used only as subscripts to identify terms, but in the second example above, it enters into the computation of the terms directly. In the next example it is used both ways.

(3) Suppose the following payments are to be made: a_1 at the end of the first period, a_2 at the end of the second period, and so on until a_n at the end of the nth period. At a fixed interest rate r, the **present value** of the payments is

$$\frac{a_1}{1+r} + \frac{a_2}{(1+r)^2} + ... + \frac{a_n}{(1+r)^n} = \sum_{i=1}^{n} \frac{a_i}{(1+r)^i}.$$

(4) Write $1 - 1/3 + 1/5 - 1/7 + 1/9 - 1/11$ using summation notation.

(5) If the summation index does not appear in the terms of the summation, then a fixed value is being added up: $\sum_{i=1}^{n} c = nc$.

Sometimes the limits of a summation are not indicated, if it is clear from context what the summation limits should be, and if we want to keep the expressions tidy and readable, e.g., we might write $\sum_i x_i$ without indicating the first or last values of i.

Rules for Working with the Summation Notation

The summation notation allows us to write, in a compact manner, expressions involving a large number of sums. They are also easily manipulable. There are essentially only two rules:

$$(i) \sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i, \qquad (ii) \sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

Examples

1. $\sum_{i=1}^{n}(x_i - \bar{x}) = 0$ where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$.
   Proof: $\sum_{i=1}^{n}(x_i - \bar{x}) = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \bar{x} = n\bar{x} - n\bar{x} = 0$.
   This shows that deviations from sample mean always sum to zero.
2. $\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \sum_{i=1}^{n}(x_i - \bar{x})y_i = \sum_{i=1}^{n}(y_i - \bar{y})x_i$
   Proof:

$$\begin{aligned}
\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) &= \sum_{i=1}^{n}(x_i - \bar{x})y_i - \sum_{i=1}^{n}(x_i - \bar{x})\bar{y} \\
&= \sum_{i=1}^{n}(x_i - \bar{x})y_i - \bar{y}\sum_{i=1}^{n}(x_i - \bar{x}) \\
&= \sum_{i=1}^{n}(x_i - \bar{x})y_i
\end{aligned}$$

The proof for $\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \sum_{i=1}^{n}(y_i - \bar{y})x_i$ is similar. You cannot drop both $\bar{x}$ and $\bar{y}$ from $\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$ unless one of the sample means is zero.

Some Useful Formulas Involving Summations

1. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.
2. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.
3. $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$.
4. Arithmetic Series: $\sum_{i=1}^{n}(a + (i-1)d) = a + (a+d) + (a+2d) + ... + (a + (n-1)d)$

$$\begin{aligned}
\sum_{i=1}^{n}(a + (i-1)d) &= na + d\sum_{i=1}^{n} i - nd \\
&= na + \frac{n(n+1)d}{2} - nd \\
&= na + \frac{n(n-1)d}{2} \\
&= \frac{n(\text{first term} + \text{last term})}{2}
\end{aligned}$$

5. Geometric Series: $\sum_{i=1}^{n} ar^{i-1} = a + ar + ar^2 + ...ar^{i-1}$.
   Let $S_n = \sum_{i=1}^{n} ar^{i-1}$. We have

$$S_n - rS_n = \sum_{i=1}^{n} ar^{i-1} - \sum_{i=1}^{n} ar^{i} = a(1 - r^n)$$

$$\therefore \quad S_n = \frac{a(1 - r^n)}{1 - r}$$

Double Summations

Suppose we have a rectangular array of numbers

$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{matrix}$$

and we wish to add up all the elements of this array. We could add up each row, then add the results of those m summations, i.e.,

$$S = \sum_{j=1}^{n} a_{1j} + \sum_{j=1}^{n} a_{2j} + \cdots + \sum_{j=1}^{n} a_{mj} = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \right)$$

or we could first add the columns, then add the results of those summations:

$$S = \sum_{i=1}^{m} a_{i1} + \sum_{i=1}^{m} a_{i2} + \cdots + \sum_{i=1}^{m} a_{in} = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} \right).$$

The parentheses make it clear which summation is done first, but conventionally leave out the parentheses and write

$$\sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij}$$

with the understanding that the inner summation is done first, and repeated for each value of the index of the outer summation. The "row then column" and "column then row" summations above show that the order of summation can be swapped, i.e.,

$$\sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}.$$

In evaluating a double summation, any term that does not depend on the inner summation index is fixed, as far as the inner summation is concerned.

Examples

1. $\sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_j = \sum_{i=1}^{n}\left(x_i \sum_{j=1}^{n} y_j\right) = \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{j=1}^{n} y_j\right).$ *Exercise* Verify this for $n = 3$.
2. $\sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_{ij} = \sum_{i=1}^{n}\left(x_i \sum_{j=1}^{n} y_{ij}\right).$ We cannot simplify this further.

Sometimes the limits of an inner summation depend on the index of the outer summation, e.g.,

$$\sum_{i=1}^{n}\sum_{j=1}^{i} a_{ij} = (a_{11}) + (a_{21} + a_{22}) + (a_{31} + a_{32} + a_{33}) + ... + (a_{n1} + a_{n2} + ... + a_{nn}).$$

In this case, we cannot interchange the order of summation; the expression $\sum_{j=1}^{i}\sum_{i=1}^{n} a_{ij}$ makes no sense!

Exercise Show that $\sum_{i=1}^{n}\sum_{j=1}^{i} a_{ij} = \sum_{j=1}^{n}\sum_{i=j}^{n} a_{ij}$.