Approximating Irrational Numbers with Newton's Method

Objective:
Explore Newton's Method for determining zeros of functions

a) How it works.
b) When it works.
c) When it fails

Background:
A zero of a function f is a value of x where the graph of $y = f(x)$ crosses the x-axis. Newton's Method gives us a way of approximating the zeros by:

a) Using an initial value (a guess) to find the next value.
b) Calculating the next value according to the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

c) This process is repeated (that is, iterations are performed) until an approximation is "close enough" to a predetermined tolerance.

It is important to realize that the above process fails if the sequence of x's does not approach the zero of $f(x)$, or if an x value occurs for which the derivative of $f(x)$ is zero.

Procedure:
Newton's Method can be used to approximate irrational numbers. For example, here is way to approximate $\sqrt{7}$, accurate to 15 decimal places.

First an equation which has $\sqrt{7}$ as a solution is needed. The most simple is $f(x) = x^2 - 7$.
In this case, $f'(x) = 2x$.

The recursion formula is

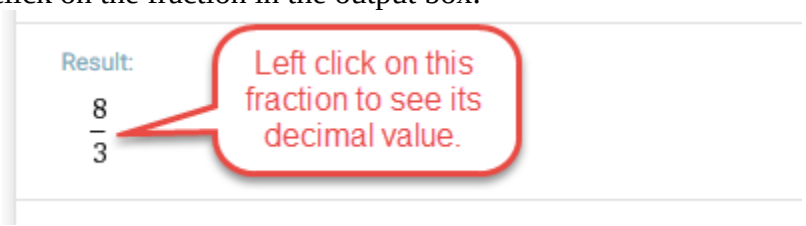
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n)^2 - 7}{2(x_n)}$$

An initial guess of $x_1 = 3$ will be used.

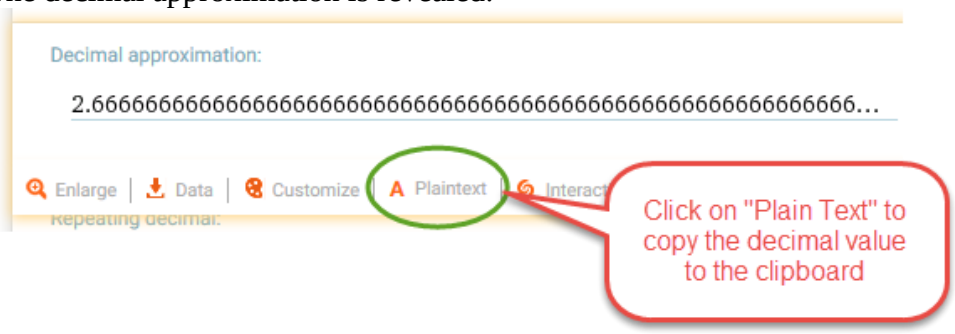
Using Wolfram Alpha:

1. Enter the first iteration into W.A.'s text box:

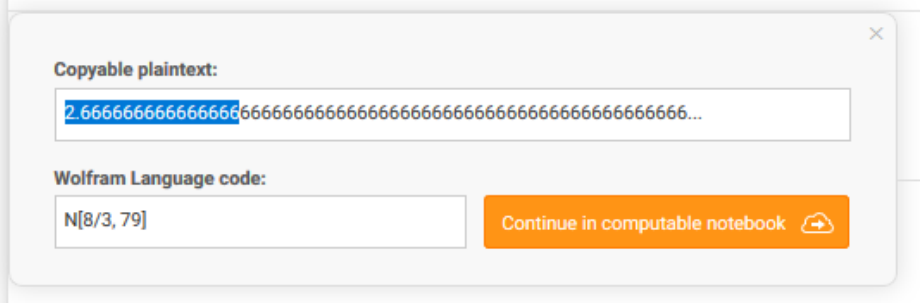
2. The result is a rational number, so W.A. outputs it as a fraction. To convert the fraction to a decimal, click on the fraction in the output box.



3. The decimal approximation is revealed:

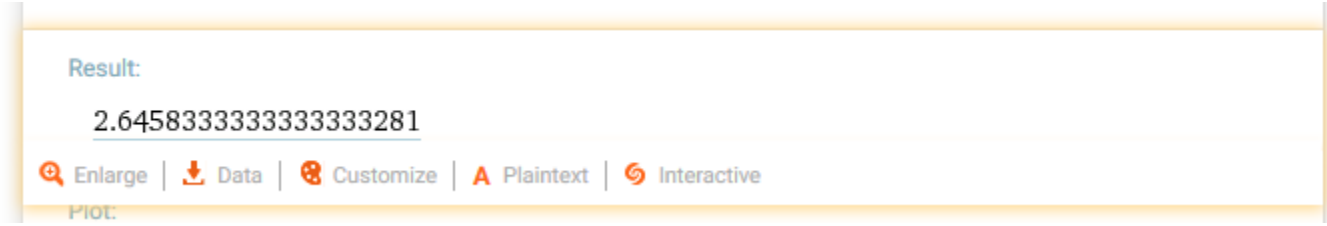


4. Highlight 15 decimal places and press Ctrl+c to copy the value to the clipboard.



5. Use the back button of your browser to navigate back and paste (Ctrl+v) the decimal value in the text box:

6. This time the decimal value is given. Copy and Paste this value to the text box and repeat for several iterations:



We are able to approximate $\sqrt{7}$ to 15 digits using 5 iterations of Newton's Method.

X_1=3
X_2= 2.666666666666666
X_3= 2.645833333333333
X_4= 2.645751312335958
X_5= 2.645751311064590
Actual value (computed in Wolfram Alpha): $\sqrt{7} \approx \boxed{2.645751311064590}3905016157536392604257102591830824501803 6$ Actual first 15 digits

Lab Report

a) Use Wolfram Alpha to investigate the zeros of the following functions using the given "initial guess." Your report should include a discussion of whether or not the process converged. If the process did not converge, why not? Would another initial guess have been better? If so, what initial guess would be better? Include graphs of these functions as well as the Newton's Method iterations along with the derivatives of the functions.

Function	**Initial Guess**
$f(x) = \sqrt{2x+3} - \sqrt{x+5}$	$x_1 = 4$
$g(x) = x^{\frac{1}{5}}$	$x_1 = 1$
$h(x) = x^5 - 5x + 1$	$x_1 = 1$

b) Approximate the value of π to 25 decimal places using Newton's Method in Wolfram Alpha. Use the sine function with an initial guess of 3. Discuss why the sine function is chosen as the appropriate function to use in this context. Include the graph of this function as well as the Newton's Method iterations.