

Mastering algebra is an important requisite of understanding calculus. The motivation for this review is to refresh algebraic concepts that might not have been encountered for some time, and to try to correct any algebraic misconceptions before delving into calculus. Inside this review are numerous examples, as well as some practice problems at the very end.

The Distributive Property

The distributive property states that

$$a(b + c) = ab + ac,$$

which implies

$$(b + c)a = ab + ac,$$

by commutativity of multiplication. We can use the distributive property to either multiply terms through, or factor terms out. For instance, if we are confronted with $13x + 4x$, we can use the distributive property to factor out the x , and find that

$$13x + 4x = (13 + 4)x = 17x.$$

This works exactly the same for a more complicated example, such as $13yx + 4x$. We can apply the distributive property to find that

$$13yx + 4x = (13y + 4)x.$$

Similarly, if we have $12(x + 2)$, the distributive property tells us that we can multiply through by 12, and find that

$$12(x + 2) = 12x + 24.$$

We can also apply the distributive property multiple times; for instance, to arrive at the important result of factoring a quadratic polynomial.

$$(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd.$$

In a similar fashion, by writing $c = d + e$ we find that

$$a(b + d + e) = ab + ad + ae.$$

Using this idea we can apply the distributive property to expressions with an arbitrary number of terms.

Common Pitfalls

1. *Not Applying the Distributive Property:* The simplest mistake one can make is to expand powers in the following way

$$(x + y)^2 = x^2 + y^2.$$

This analysis completely ignores the distributive property - we should find that

$$(x + y)^2 = x^2 + 2xy + y^2.$$

It is noteworthy that these two expressions are only equal if either x or y is zero, in which the problem is reduced to one in which we do not need the distributive property.

2. *Not Distributing Negative Signs:* It is important to remember to carry through negative signs when using the distributive property. For instance,

$$-(x + 1) = -x - 1.$$

Note that the negative sign is “carried” to both terms. Similarly,

$$-h(x + 1) = -hx - h,$$

where once again the second term is also a negative h . Similarly, we find that

$$h(x - 1) = hx - h,$$

where the second term is negative, because we had a negative 1.

3. *Distributing n^{th} Powers:* It may seem tempting to directly apply the distributive property to expressions such as

$$2(x + 1)^2,$$

by writing that

$$2(x + 1)^2 = (2x + 2)^2.$$

This, however, is incorrect. If we expand both expressions we find that

$$2(x + 1)^2 = 2(x + 1)(x + 1) = 2(x^2 + 2x + 1) = 2x^2 + 4x + 2,$$

and

$$(2x + 2)^2 = (2x + 2)(2x + 2) = 4(x + 1)(x + 1) = 4x^2 + 8x + 4,$$

so clearly the two expressions are not equal. The above analysis does, however, suggest how we could properly rewrite the expression using the distributive property.

$$2(x + 1)^2 = 2(x + 1)(x + 1) = \sqrt{2} \cdot \sqrt{2}(x + 1)(x + 1) = (\sqrt{2}x + \sqrt{2})(\sqrt{2}x + \sqrt{2}) = (\sqrt{2}x + \sqrt{2})^2$$

Although we are able to rewrite the expression in this way, it is generally not worth the effort required to do so. One is better off simply leaving an expression of the form $2(x + 1)^2$ as it is. Similarly, factoring coefficients out of such an expression such as $(2x + 2)^2$ is generally not worthwhile.

4. *Improperly Separating Fractions:* Suppose we have the fraction

$$\frac{x + 1}{x - 3}$$

One might be tempted to say that

$$\frac{x + 1}{x - 3} = \frac{x + 1}{x} - \frac{x + 1}{3},$$

but this would not be correct, because we cannot separate the terms in the denominator. As a simple example, note that

$$\frac{1}{4} = \frac{1}{2 + 2} \neq \frac{1}{2} + \frac{1}{2} = 1.$$

Nevertheless, we can separate the terms in the numerator using the distributive property, so

$$\frac{x + 1}{x - 3} = \frac{1}{x - 3}(x + 1) = \frac{x}{x - 3} + \frac{1}{x - 3}.$$

is a valid statement.

Arithmetic Operators

The arithmetic operators are $+$, $-$, \times , \div (addition, subtraction, multiplication, and division. If we have an equation, say $x = y$, we can perform any arithmetic operation involving variables or constants (other than dividing by 0) to both sides of the equation without changing the truth of the equation (although we can multiply by 0, we gain no information from doing so). Given that $x = y$ it follows that

$$\frac{x + 2 - p}{5} = \frac{y + 2 - p}{5},$$

by applying the same arithmetic operations to both sides of the equation. In general we do not wish to use the arithmetic operations to complicate the equation; we want to use them in a way to isolate and solve for some variable. For instance, if we have $2x + 1 = 2$, we can first subtract 1 from both sides to find

$$2x = 1.$$

Then we can divide both sides of the equation by 2 to find

$$x = \frac{1}{2}.$$

We can also use this technique to solve systems of equations. Suppose we have that

$$\begin{aligned} 2x + y &= 5 \\ x - y &= 2. \end{aligned}$$

This first equation tells us that $2x + y$ and 5 are equal, and since we can apply equivalent arithmetic operations to both sides of an equation, we can add $2x + y$ to the left-hand side of the second equation, and add 5 to the right-hand side of the second equation. Thus, we find that

$$\begin{aligned} x - y + 2x + y &= 2 + 5 \\ 3x &= 7 \\ x &= 7/3. \end{aligned}$$

Having solved for x , we can substitute it into either of the equations to find $y = 1/3$.

Common Pitfalls

1. *Dividing by the Variable:* If you are faced with an equation that looks like

$$x(2x + 3) = x,$$

it may seem very tempting to divide both sides of the equation by x . However, this is not a valid operation if $x = 0$. Thus, dividing both sides of the equation by x carries the assumption that $x \neq 0$. Dividing both sides of the equation by x we find $2x + 3 = 1$, so our *only* solution is $x = -1$.

This is untrue. While $x = -1$ is a solution, it is not the *only* solution ($x = 0$ is also solution). Thus, when you divide both sides of the equation by x , you lose one of the solutions to the equation, so you find a single solution rather than two solutions, which is incorrect (because you are assuming $x \neq 0$, but $x = 0$ is actually a solution).

In general, if

$$a \cdot b = 0$$

then one of the following is true: $a = 0$, $b = 0$, or $a = b = 0$. Thus, when one is faced with an equation like

$$x(x + 2) = 0$$

there are two solutions: $x = 0$, and $x + 2 = 0$, which implies $x = -2$.

2. *Not Performing an Operation to All Terms of an Equation:* Suppose that

$$\eta(x + 4) - x = \gamma.$$

In this case, one might try to divide both sides of the equation by η to find

$$(x + 4) - x = \frac{\gamma}{\eta}.$$

This is incorrect, because dividing both sides of the equation by η does not simply move η to the right side of the equation. Performing such an operation would actually yield

$$x + 4 - \frac{x}{\eta} = \frac{\gamma}{\eta},$$

which is not extremely useful in solving the equation. A better strategy would be to multiply through by η using the distributive property to find

$$\eta x + 4\eta - x = \gamma.$$

From there the x terms could be combined, yielding

$$x(\eta - 1) + 4\eta = \gamma.$$

Next we would subtract 4η from both sides and find

$$x(\eta - 1) = \gamma - 4\eta.$$

Finally, we would divide by $(\eta - 1)$ completing the solution for x

$$x = \frac{\gamma - 4\eta}{\eta - 1}.$$

Multiplying by 1

The multiplicative identity is 1, which means that for any number a , we find

$$a \cdot 1 = a$$

Thus, unlike the arithmetic operators, which we must apply to both sides of the equation, we can multiply a single entity by 1 without changing our equation at all. For instance, if we need to solve

$$\frac{x}{7} + \frac{x}{2} = 1,$$

we can multiply both of the fraction terms by 1, in order to create a common denominator. Thus

$$\frac{x}{7} + \frac{x}{2} = \frac{x}{7} \cdot 1 + \frac{x}{2} \cdot 1 = \frac{x}{7} \cdot \frac{2}{2} + \frac{x}{2} \cdot \frac{7}{7} = \frac{2x}{14} + \frac{7x}{14} = \frac{2x + 7x}{14}.$$

Up to this point, we've used the trick of multiplying by 1 in order to create a common denominator. Now we can apply the distributive property to factor out the x , and then perform arithmetic operations to find a solution. We find that

$$\begin{aligned}\frac{2x + 7x}{14} &= 1 \\ \frac{(2 + 7)x}{14} &= 1 \\ \frac{9x}{14} &= 1 \\ 9x &= 14 \\ x &= \frac{14}{9}.\end{aligned}$$

Common Pitfalls

1. *Improperly Canceling Terms of a Fraction:* Suppose you are faced with a fraction containing polynomial terms, such as

$$\frac{x}{x+1}.$$

In this situation, one might see an x in the numerator and denominator of the function, and feel tempted to cancel them, yielding either

$$\frac{\cancel{x}}{\cancel{x}+1} = \frac{0}{0+1} = \frac{0}{1} = 0,$$

or

$$\frac{\cancel{x}}{\cancel{x}+1} = \frac{1}{1+1} = \frac{1}{2}.$$

Both of these are incorrect. In order to *cancel* an x from both the numerator and the denominator, it would have to be multiplying *all* of the terms in both the numerator and denominator, as follows

$$\frac{2x}{x^2 + x}.$$

Above we can factor x from the terms in the denominator, and then *cancel* a factor of x from the numerator and denominator as follows

$$\frac{2x}{x^2 + x} = \frac{2x}{x(x+1)} = \frac{x}{x} \cdot \frac{2}{x+1} = \frac{2}{x+1}.$$

Note that the expression

$$\frac{2x}{x^2 + x}$$

is not defined for $x = 0$. When we write

$$\frac{x}{x} = 1$$

we are assuming that $x \neq 0$. Thus, the expressions

$$\frac{2x}{x^2 + x} \quad \text{and} \quad \frac{2}{x+1}$$

are only equal when $x \neq 0$.

Similarly, if we have a fraction multiplied by a constant, we need to distribute the constant to all terms within the numerator of the fraction, so

$$2 \cdot \frac{x+1}{x} = \frac{2(x+1)}{x} = \frac{2x+2}{x}.$$

Recognizing Constants

When solving an equation, it is of the utmost importance to be cognizant of which variable you are solving for. Suppose you have the equation $x + 2y - z = 14$. If you are solving for x , then you can consider y and z to be constants (but be careful, because you can only divide by y or z if you know that they are constants that do not equal 0). Similarly, if you were solving for y , then x and z would be constants, with the same disclaimer as before.

Now what about a situation with more complicated *looking* constants? Suppose

$$2x - \frac{1 - e^{\alpha\tau}}{y + 1} = 32$$

is the equation you need to solve, and you need to solve for x . Since you are solving for x , that means that α , τ , and y are all constants. As long as you are careful not to divide by 0, you can treat these constants the same as any number. Similarly, any expression that is a combination of these constants, such as

$$\frac{1 - e^{\alpha\tau}}{y + 1}$$

can simply be treated as a single constant. Upon recognizing that fact, the equation we are faced with is just

$$2x - (\text{some constant}) = 32.$$

Now we can simply use our arithmetic operators, and add *some constant* to both sides of the equation, to find

$$2x = 32 + \frac{1 - e^{\alpha\tau}}{y + 1}.$$

Afterwards, all we need to do is divide both sides by 2, to find the result

$$x = \frac{1}{2} \left(32 + \frac{1 - e^{\alpha\tau}}{y + 1} \right).$$

As you can see, once you recognize that the very complicated looking term $\frac{1 - e^{\alpha\tau}}{y + 1}$ is just a constant that can be manipulated as a single entity, the problem becomes relatively simple. Just as we were able to manipulate the complicated constant term as a single term, we can manipulate complicated terms consisting of variables as a single entity as well. In the following example

$$2x = 32 + \frac{1 - e^{\alpha\tau}}{y + 1}x$$

we can subtract by the term $\frac{1 - e^{\alpha\tau}}{y + 1}x$ to find

$$2x - \frac{1 - e^{\alpha\tau}}{y + 1}x = 32.$$

Now we apply the distributive property to factor out the x , yielding

$$\left(2 - \frac{1 - e^{\alpha\tau}}{y + 1} \right)x = 32.$$

Finally we can divide by the even more complicated term, and find

$$x = \frac{32}{2 - \frac{1 - e^{\alpha\tau}}{y + 1}}.$$

Similarly, if we have a term consisting of multiple variables such as

$$x = p(2x + 1)x,$$

we can manipulate it as a single entity yielding

$$x - p(2x + 1)x = 0.$$

Now applying the distributive property we find

$$x(1 - p(2x + 1)) = 0.$$

This is an equation with two solutions $x = 0$, and $1 - p(2x + 1) = 0$, or $x = \frac{1}{2}(\frac{1}{p} - 1)$. It is left to the reader to verify these are indeed solutions.

General Strategy

In order to solve algebraic equations, one must apply the above tools in the proper order, and avoid the common pitfalls along the way. The general strategy to solving an algebraic equation is as follows

1. *Separate the variable and constant terms.* The first step is to apply arithmetic operators in order to move all terms containing the variable being solved for to one side of the equation, and the remaining constant terms to the other. After this process is completed, if possible, simplify the expressions (such as canceling a 1 and -1 on the same side of the equation).
2. *Factor the variable from constant terms, if necessary.* If there are multiple terms containing the variable, use the distributive property to factor out the variable from the constants. Depending on the form of the equation, more complicated factoring schemes may need to be used (such as $(a + b)(c + d) = ac + ad + bc + bd$).
3. *Divide by any constant terms that resulted from factoring, to isolate the variable.* If you were able to factor out all instances of the variable from the constant terms in the previous step, then divide by the constant in order to isolate the variable.
4. *If the equation has multiple terms multiplied together which equal 0, check for solutions to each of the terms equalling 0.* It may turn out that your equation looks something like $x(x - 2) = 0$ after it is simplified. Remember that setting either of the terms on the left equal to zero provides a solution, so there are 2 solutions, not just one ($x = 0$ and $x = 2$).

It should be emphasized that this is a general strategy, but it is not an algorithm that may be followed blindly in order to solve an equation. Thus, it is only with practice, and an understanding of how to utilize the tools presented above that one can master solving algebraic equations. Having read the above examples, try to solve the following equations for x .

$$p^2(2x + 4) + e^{\alpha\tau} = 1$$

$$p(2x - 17) = 4q + x$$

$$x = \frac{x}{x + p(1 - x)}$$