

# Irrational Numbers

## A. Rational Numbers

1. Before we discuss irrational numbers, it would probably be a good idea to define rational numbers.

### Definition: Rational Number

A rational number is a number that can be written as a ratio (i.e. fraction)  $\frac{a}{b}$ , where  $a$  and  $b$  are integers (and  $b \neq 0$ ).

2. Examples of rational numbers:

a)  $\frac{2}{3}$

b)  $\frac{-5}{2}$

c)  $\frac{7.2}{1.3}$        $\frac{7.2}{1.3}$  is a rational number because it is equivalent to  $\frac{72}{13}$ .

d) 6      6 is a rational number because it is equivalent to  $\frac{6}{1}$ .

e) -4      -4 is a rational number because it is equivalent to  $\frac{-4}{1}$ .

f) 0.2      0.2 is a rational number because it is equivalent to  $\frac{2}{10}$  (or  $\frac{1}{5}$ ).

g)  $0.\bar{3}$        $0.\bar{3}$  is a rational number because it is equivalent to  $\frac{1}{3}$ .

*Note: Any terminating decimal (such as 0.2, 0.75, 0.3157) is a rational number. Any repeating decimal (such as  $0.\bar{3}$ ,  $0.4\bar{5}$ ,  $0.37\bar{214}$ ) is a rational number. (There are mathematical ways of converting repeating decimals to fractions which will not be covered in this workshop.)*

*Can you think of any numbers that are not rational numbers?...*

## B. Irrational Numbers

1. The technical definition of an irrational number is that it is a “real number which is not a rational number.” So what does an irrational number look like?

An irrational number is a nonterminating, nonrepeating decimal.

2. A few examples of irrational numbers are  $\pi$ ,  $\sqrt{2}$ , and  $\sqrt{3}$ . (In fact, the square root of any prime number is irrational. Many other square roots are irrational as well.)

The values of  $\pi$ ,  $\sqrt{2}$ , and  $\sqrt{3}$  are shown below to 50 decimal places. (Notice the nonrepeating nature of the numbers.)

$$\pi \approx 3.14159265358979323846264338327950288419716939937510\dots$$

$$\sqrt{2} \approx 1.41421356237309504880168872420969807856967187537694\dots$$

$$\sqrt{3} \approx 1.73205080756887729352744634150587236694280525381038\dots$$

We will now focus more on how to calculate and simplify square roots.

## C. Evaluating Rational Square Roots

1. Recall how to find square roots:

a)  $\sqrt{9} = 3$ , since  $3 \cdot 3 = 9$

b)  $\sqrt{36} = 6$ , since  $6 \cdot 6 = 36$

c)  $\sqrt{\frac{49}{100}} = \frac{7}{10}$ , since  $\frac{7}{10} \cdot \frac{7}{10} = \frac{49}{100}$

*Note: If we solve an equation such as  $x^2=25$ , we take the square root of both sides and obtain a solution of  $x=\pm 5$ . However, the “ $\sqrt{\quad}$ ” symbol denotes the principle square root and represents only the positive square root. Therefore we say, for example, that  $\sqrt{25}=5$ , NOT  $\sqrt{25}=\pm 5$ .*

2. Evaluate the following without a calculator. Give exact answers:

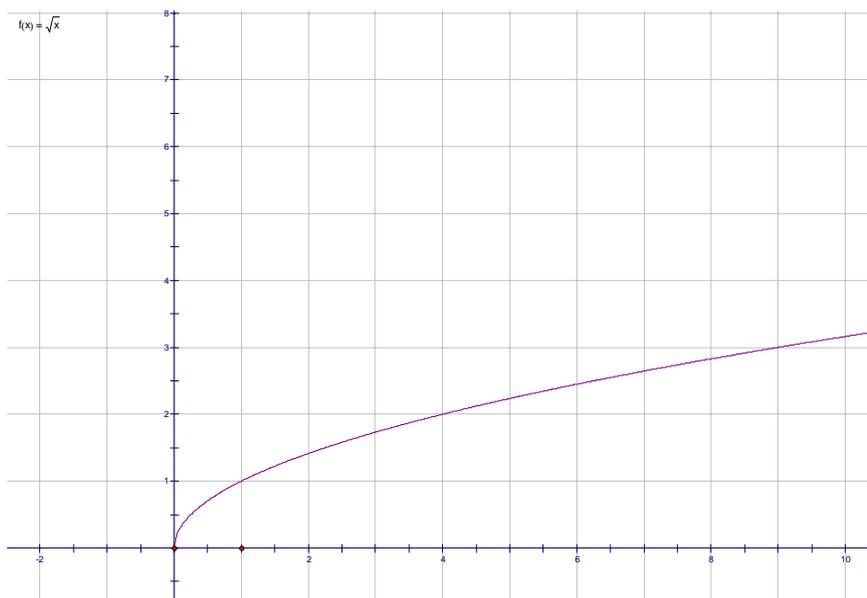
a)  $\sqrt{64} = \underline{\hspace{2cm}}$       b)  $\sqrt{25} = \underline{\hspace{2cm}}$       c)  $\sqrt{\frac{16}{81}} = \underline{\hspace{2cm}}$

## D. Approximating Square Roots

1. We will first use the graph of  $y = \sqrt{x}$  to approximate square roots. To graph  $y = \sqrt{x}$ , let us first make a chart with a few  $x$  and  $y$  values:

$x$	$y = \sqrt{x}$
0	0
1	1
4	2
9	3

If we plot the above points and then draw a smooth curve through them, we obtain the following graph of  $y = \sqrt{x}$ :



By looking at the graph, we can get a general idea of the value of some irrational square roots:

- Use the graph to estimate the value of  $\sqrt{5}$  to the nearest tenth. \_\_\_\_\_
- Use the graph to estimate the value of  $\sqrt{2}$  to the nearest tenth. \_\_\_\_\_
- Use the graph to estimate the value of  $\sqrt{8}$  to the nearest tenth. \_\_\_\_\_
- Compare the values you obtained for  $\sqrt{2}$  and  $\sqrt{8}$ . Do you notice any relationship between them? \_\_\_\_\_.
- Fold the graph so that the point  $(8, \sqrt{8})$  is on top of the point  $(8, 0)$ . After unfolding your paper, look to see where the crease hits the graph. Does this verify the relationship that you found in part (d)? \_\_\_\_\_

## 2. Mental Estimation of Square Roots

Suppose that we want to estimate the value of  $\sqrt{50}$ . It may not be reasonable to extend the above graph to an  $x$  value of 50 so approximate this square root. To estimate square roots mentally, it is useful to refer to a list of perfect squares:

$$\begin{aligned}1^2 &= 1 \\2^2 &= 4 \\3^2 &= 9 \\4^2 &= 16 \\5^2 &= 25 \\6^2 &= 36 \\7^2 &= 49 \\8^2 &= 64 \\9^2 &= 81 \\10^2 &= 100 \\&\vdots\end{aligned}$$

<u>Perfect Squares</u>
<b>1</b>
<b>4</b>
<b>9</b>
<b>16</b>
<b>25</b>
<b>36</b>
<b>49</b>
<b>64</b>
<b>81</b>
<b>100</b>
<b>⋮</b>
<i>(continue as needed)</i>

If we want to estimate the value of  $\sqrt{50}$ , for example, we can use the perfect squares list in this way: The number 50 is between the perfect squares 49 and 64. Therefore,  $\sqrt{50}$  is between  $\sqrt{49}$  and  $\sqrt{64}$ . So  $\sqrt{50}$  has a value between 7 and 8.

Examples: Estimate the following square roots, using the method from above.

a)  $\sqrt{90}$

The number 90 is between the perfect squares \_\_\_\_\_ and \_\_\_\_\_. Therefore,  $\sqrt{90}$  is between  $\sqrt{\text{_____}}$  and  $\sqrt{\text{_____}}$ . So  $\sqrt{90}$  has a value between \_\_\_\_\_ and \_\_\_\_\_.

b)  $\sqrt{23}$

The number 23 is between the perfect squares \_\_\_\_\_ and \_\_\_\_\_. Therefore,  $\sqrt{23}$  is between  $\sqrt{\text{_____}}$  and  $\sqrt{\text{_____}}$ . So  $\sqrt{23}$  has a value between \_\_\_\_\_ and \_\_\_\_\_.

### **E. Using a Calculator to Evaluate Square Roots**

1. Find the  $\sqrt{\quad}$  key on your calculator. (On many calculators, including the TI-83, it is “behind” the  $x^2$  key.) To find the decimal value of  $\sqrt{10}$  on a TI-83, for example, you should first press the “2<sup>nd</sup>” key, then press the “ $x^2$ ” key, then enter 10, then press the “Enter” key. You should get a value of 3.1623 (if you round to the nearest ten-thousandth).
2. Enter the following values into your calculator and make sure that you get the same answers as those listed here. (Ask your workshop instructor if you need assistance.) Irrational answers have been rounded to the nearest ten-thousandth.

a) $\sqrt{50}$	Answer: $\sqrt{50} \approx 7.0711$
b) $\sqrt{676}$	Answer: $\sqrt{676} = 26$
c) $5\sqrt{7}$	Answer: $5\sqrt{7} \approx 13.2288$
d) $\frac{2\sqrt{3}}{3}$	Answer: $\frac{2\sqrt{3}}{3} \approx 1.1547$

### **F. Simplifying Square Roots**

1. What if we are asked to simplify a square root such as  $\sqrt{75}$ ? This square root can not be written as a rational number, but we can still write it in a simpler form.
2. One method of simplifying square roots will be shown here. To use this method, we must again use our list of perfect squares:

$$\begin{aligned}1^2 &= 1 && \text{*See note below} \\2^2 &= 4 \\3^2 &= 9 \\4^2 &= 16 \\5^2 &= 25 \\6^2 &= 36 \\7^2 &= 49 \\8^2 &= 64 \\9^2 &= 81 \\10^2 &= 100 \\&\vdots\end{aligned}$$

<b><u>Perfect Squares*</u></b>
<b>4</b>
<b>9</b>
<b>16</b>
<b>25</b>
<b>36</b>
<b>49</b>
<b>64</b>
<b>81</b>
<b>100</b>
<b>⋮</b>
<i>(continue as needed)</i>

*\*The number 1 is a perfect square, but we have not included it in our list at the right because it is not helpful to us in simplifying square roots (This should make more sense as you read the examples below.)*

3. Detailed Examples:

a) Write  $\sqrt{75}$  in simplest radical form.

(i) First, find the largest perfect square (see our list) that is a factor of 75. This number is 25.

(ii) Write  $\sqrt{75}$  as a product of two factors, using the number from step (i), and then simplify

$$\begin{aligned}\sqrt{75} &= \sqrt{25} \cdot \sqrt{3} \\ &= 5\sqrt{3}\end{aligned}$$

$$\boxed{\text{So } \sqrt{75} = 5\sqrt{3}.$$

b) Write  $\sqrt{90}$  in simplest radical form.

(i) First, find the largest perfect square (see our list) that is a factor of 90. This number is 9.

(ii) Write  $\sqrt{90}$  as a product of two factors, using the number from step (i), and then simplify

$$\begin{aligned}\sqrt{90} &= \sqrt{9} \cdot \sqrt{10} \\ &= 3\sqrt{10}\end{aligned}$$

$$\boxed{\text{So } \sqrt{90} = 3\sqrt{10}.$$

*Note: It is not helpful to write  $\sqrt{10}$  as  $\sqrt{5} \cdot \sqrt{2}$ , since both  $\sqrt{5}$  and  $\sqrt{2}$  are irrational and can not be simplified any further.*

c) Write  $\sqrt{72}$  in simplest radical form.

(i) First, find the largest perfect square (see our list) that is a factor of 72. This number is 36.

(iii) Write  $\sqrt{72}$  as a product of two factors, using the number from step (i), and then simplify

$$\begin{aligned}\sqrt{72} &= \sqrt{36} \cdot \sqrt{2} \\ &= 6\sqrt{2}\end{aligned}$$

$$\boxed{\text{So } \sqrt{72} = 6\sqrt{2}.$$

What if we had chosen a smaller factor of 72 from the list of perfect squares, such as 9? We could still follow the same steps, but it would take a little longer to simplify completely:

$$\begin{aligned}
 \sqrt{72} &= \sqrt{9} \cdot \sqrt{8} \\
 &= 3\sqrt{8} && \text{Notice that } \sqrt{8} \text{ can be simplified further.} \\
 &= 3\sqrt{4} \cdot \sqrt{2} \\
 &= 3 \cdot 2 \cdot \sqrt{2} \\
 &= 6\sqrt{2}
 \end{aligned}$$

So $\sqrt{72} = 6\sqrt{2}$ .
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So if we choose the largest factor from the Perfect Squares list, we complete the simplification in one step. If the radicand (the number under the square root) is large, it may be easier to simplify in several steps rather than to extend our list of perfect squares.

- d) Although we have discussed this method of simplifying radicals for irrational square roots in the previous examples, it works for rational square roots as well, and is particularly valuable if we need to find the square root of a large number without a calculator.

For example, suppose that we want to write  $\sqrt{3600}$  in simplest radical form. If we do not have a calculator (and if the square root is not already apparent), let us find an obvious perfect square which is a factor of 3600 (instead of extending our perfect squares list) – say 100.

$$\begin{aligned}
 \sqrt{3600} &= \sqrt{100} \cdot \sqrt{36} \\
 &= 10 \cdot 6 \\
 &= 60
 \end{aligned}$$

So $\sqrt{3600} = 60$ .
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3. Examples: Write the following in simplest radical form.

a)  $\sqrt{32} =$  \_\_\_\_\_

b)  $\sqrt{27} =$  \_\_\_\_\_

c)  $\sqrt{14} =$  \_\_\_\_\_

d)  $\sqrt{48} =$  \_\_\_\_\_

e)  $\sqrt{120} =$  \_\_\_\_\_

f)  $\sqrt{1280} =$  \_\_\_\_\_