Teaching Irrational Numbers Through Trigonometry*

Sameen Ahmed Khan

An introduction to irrational numbers is done using basic trigonometry, generally taught in pre-university courses. The irrational sets of trigonometric ratios of rational angles are discussed in detail along with two major theorems and their proofs. Algebraic numbers along with transcendental numbers are also covered.

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1. Introduction

High school mathematics introduces the concept of irrational numbers, and as an example proves that $\sqrt{2}$ is irrational using *proof by contradiction*. A little bit more is done by stating that in general, for any prime p, $\sqrt{p}$ is also irrational. The textbooks and the instructors end the topic by giving some examples of numbers such as π and e, which are not rational. Coverage to the irrational numbers can be enriched through the use of basic trigonometry, generally taught in the pre-university courses. The sine or cosine of a rational number of degrees (if in radians, it is a rational multiple of π) are irrational numbers. The only exceptions are $\cos\alpha, \sin\alpha \in \{0, \pm\frac{1}{2}, \pm 1\}$. We shall look at the related theorems with proofs based on the results from elementary trigonometry.

2. Irrationality of Trigonometric Ratios

Keywords
Trigonometric ratios, algebraic numbers, irrational numbers, transcendental numbers, Niven's theorem, Paolillo–Vincenzi theorem, Ram Murty–Kumar Murty theorem, Ailles rectangle.

The arithmetic properties of trigonometric functions are a recurring topic. The contributions of several mathematicians are summarized in the following theorem published by Ivan Morton Niven in the year 1956 and widely known as *Niven's theorem*.

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Box 1. Gregory Numbers

Gregory number, named after James Gregory is defined as a number of the form

$$
\begin{aligned}
G_x &= \tan^{-1}\left(\frac{1}{x}\right), \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)x^{2k+1}},
\end{aligned}
$$

where x is an integer or a rational number. For example, for $x = 1$, $G_1 = \tan^{-1}(1) = 45°(\pi/4)$ is a Gregory number. All Gregory numbers in degrees are irrational with the exception of the pair $G_{-1} = -45°(-\pi/4)$ and $G_1 = 45°(\pi/4)$. Historically, the arctangent ($\tan^{-1}$) identities have been extensively used for calculating the value of π. The simplest case, $x = 1$ leads to

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots .$$

This series is called the Madhava–Leibniz series (also known as Gregory's series) and is a special case of a more general series expansion for the inverse tangent function, first discovered by the Indian mathematician Madhava of Sangamagrama in the 14th century. The special case was published by Gottfried Leibniz in 1676.

Theorem 1. *Niven's Theorem: The only rational values of α in the interval* $0° \leq \alpha \leq 90°$ *for which the sine of α degrees is also a rational number are* $\sin 0° = 0$, $\sin 30° = \frac{1}{2}$, $\sin 90° = 1$.

The theorem appears in the two books on irrational numbers by Niven. The theorem implies that for rational angles in degrees, the only rational values of the trigonometric ratios are $\cos\alpha, \sin\alpha \in \{0, \pm\frac{1}{2}, \pm 1\}$, $\sec\alpha, \csc\alpha \in \{\pm 1, \pm 2\}$ and $\tan\alpha, \cot\alpha \in \{0, \pm 1\}$. There are different proofs to the theorem using diverse techniques including, induction, de Moivre formulas, Chebyshev polynomials, cyclotomic polynomials, among others.

Proof. Let us assume that $2\cos(\alpha) = a/b$, where a and b are integers, $b \neq 0$ and a/b is in the reduced form with no common factors. We write the double angle identity as $2\cos(2\alpha) =$

$(2\cos(\alpha))^2 - 2$, then,

$$2\cos(2\alpha) = \frac{a^2 - 2b^2}{b^2}. \tag{1}$$

We will use the notation, $a|b$ for *a divides b*, if there is an integer c such that $b = ac$. Now, (a^2-2b^2) and b^2 have no common factors, since if p were a prime number dividing both, then $p|b^2$ implying $p|b$ and $p|(a^2 - 2b^2)$ implying $p|a$, leading to a contradiction. So, $(a^2 - 2b^2)/b^2$ is also in the reduced form. Using the double angle identity repeatedly, we construct the sequence $2\cos(\alpha)$, $2\cos(2\alpha)$, $2\cos(2^2\alpha)$, $2\cos(2^3\alpha)$, $\ldots$, $2\cos(2^k\alpha)$. The denominators in the sequence grow rapidly as $b^{(2^k)}$. The cos function is periodic with period 360°. If $\alpha = (m/n)360°$, where m and n are integers, $n \neq 0$ and m/n is in the reduced form, then the sequence $\{2\cos(2^k\alpha)\}$ may admit at most n different values. This assertion is based on the expansion of $\cos(\phi)$ in a n-th degree polynomial in $x = \cos(\phi/n)$. Any n-th degree polynomial has n solutions including the repeating solutions (if any). Hence, the phrase *at most* in the assertion (see Section 3). The presence of n solutions contradicts the growth of the denominators, and the only allowed values of b are $b = \pm 1$. With $b = \pm 1$, $\cos\alpha = a/2b = \pm a/2$. So, $a \in \{0, \pm 1, \pm 2\}$. Consequently, $\cos\alpha \in \{0, \pm\frac{1}{2}, \pm 1\}$. Same set of values apply to $\sin\alpha$, as $\sin\alpha = \cos(90° - \alpha)$.

According to Paolillo–Vincenzi theorem, If α is rational in degrees, say $\alpha = (m/n)180°$ for some rational number m/n, and $\tan^2(\alpha)$ is rational, then $\tan^2(\alpha) \in \{0, 1, \frac{1}{3}, 3\}$.

A recent proof (year 2020) of Niven's theorem is due to Bonaventura Paolillo and Giovanni Vincenzi. It is based on the periodicity of the tangent function and has some additional results.

Theorem 2. *Paolillo–Vincenzi Theorem: If α is rational in degrees, say $\alpha = (m/n)180°$ for some rational number m/n, and* $\tan^2(\alpha)$ *is rational, then* $\tan^2(\alpha) \in \{0, 1, \frac{1}{3}, 3\}$.

Proof. Let us suppose that $\tan^2(\alpha) = \tan^2(\alpha_m) = \frac{a_m}{b_m}$, ($a_m$ and b_m are positive integers and a_m/b_m is in the reduced form) is a rational number different from 0 and 1. The angle $\alpha_i = \frac{i}{n}180°$ for some positive integer i and n can be any integer different from 0. We construct a non-empty set of rational numbers as follows:

$$T_n := \left\{\tan^2(\alpha_i) \in \mathbb{Q} \setminus \{0\,,1\} : i \in \mathbb{N}\right\}\,, \tag{2}$$

where $\mathbb{N}$ is the set of natural numbers, and $\mathbb{Q}$ is the set of rational numbers. The notation, $A \setminus B$ denotes the *set minus* such that $A - B$ is the set of elements in A but not in B. Each element of T_n is of the type $\tan^2(\alpha_i) := \frac{a_i}{b_i}$ where a_i and b_i constitute a pair of positive integers such that $\frac{a_i}{b_i}$ is in the reduced form. From the set T_n we choose an element $\frac{a_k}{b_k}$ such that

$$\tan^2(\alpha_k) := \frac{a_k}{b_k}\,, \qquad (a_k + b_k) = \max\left\{(a_i + b_i) : \frac{a_i}{b_i} \in T_n\right\}\,. \tag{3}$$

From the construction of the set T_n, it follows that $b_k \neq 0$ and $a_k \neq b_k$. So, the sum of the numerator and denominator, $(a_k + b_k) \geq 3$. Likewise, by construction, $\alpha_k \neq (45^\circ + h90^\circ)$ for every integer h. Let us consider the case, $\alpha_{2k} = 2\alpha_k = 2\frac{k}{n}180^\circ$, then

$$\tan^2(\alpha_{2k}) = (\tan(2\alpha_k))^2 = \left(\frac{2\tan(\alpha_k)}{1 - \tan^2(\alpha_k)}\right)^2 = \frac{4a_kb_k}{(a_k - b_k)^2}\,, \tag{4}$$

which, by construction, is a rational number different from 0 and 1. By construction, both a_k and b_k cannot be even. If one of a_k and b_k is even then, the other is necessarily odd, implying $(a_k - b_k)$ is odd. Hence, no prime deviser of $(a_k - b_k)$ can divide $4a_kb_k$, because a_k and b_k are coprime. So, $4a_kb_k/(a_k - b_k)^2$ is a reduced fraction and hence by the preposition (3) on the sum of numerator and denominator, we have

$$4a_kb_k + (a_k - b_k)^2 = (a_k + b_k)^2 \leq (a_k + b_k)\,, \tag{5}$$

which implies $(a_k + b_k) \leq 1$, leading to a clear contradiction. So, both a_k and b_k are odd (and coprime by construction). We rewrite the reduced fraction in (4) as

$$\begin{aligned}\tan^2(\alpha_{2k}) = (\tan(2\alpha_k))^2 &= \left(\frac{2\tan(\alpha_k)}{1 - \tan^2(\alpha_k)}\right)^2 \\ &= \frac{a_kb_k}{[(a_k - b_k)^2]/4} \in T_n\,. \end{aligned} \tag{6}$$

It follows by the preposition (3), on the sum of numerator and denominator

$$a_k b_k + \frac{(a_k - b_k)^2}{4} = \frac{(a_k + b_k)^2}{4} \leq (a_k + b_k)\,, \tag{7}$$

implying that

$$\begin{aligned} (a_k + b_k)^2 &\leq 4(a_k + b_k) \\ (a_k + b_k) &\leq 4\,. \end{aligned} \tag{8}$$

So, the sums of the numerators and denominators in the preposition (3) are bounded by the inequalities

$$3 \leq (a_k + b_k) \leq 4\,. \tag{9}$$

Having established the lower and upper bounds of $(a_k + b_k)$, the next step is to see the implications of these bounds on the values of a_k and b_k respectively. By construction, $\tan^2(\alpha) = \tan^2(\alpha_m) = \frac{a_m}{b_m}$ and $\frac{a_m}{b_m} \neq 0, 1$. So, $\tan^2(\alpha_{2m}) = 4a_m b_m/(a_m - b_m)^2$ is also different from 0 and 1. As a particular case, $\tan^2(\alpha_{2m}) = a_{2m}/b_{2m} \in T_n$. By preposition (3), $(a_m + b_m)$ and $(a_{2m} + b_{2m})$ are bounded by the inequalities in (9). By construction, $(a_m, b_m) \neq (0, 1)$, $(a_m, b_m) \neq (1, 0)$, $(a_m, b_m) \neq (1, 1)$, $(a_m, b_m) \neq (2, 2)$ and $(a_m, b_m) \neq (0, 4)$. So, we examine the case $(a_m, b_m) = (1, 2)$, which implies $\tan^2(\alpha_{2m}) = 4a_m b_m/(a_m - b_m)^2 = a_{2m}/b_{2m} = 8$ leading to $(a_{2m} + b_{2m}) = 8 + 1 = 9$, which is a contradiction to the inequalities in (9). Similarly, the case $(a_m, b_m) = (2, 1)$ also leads to the same contradiction. Lastly, the cases $(a_m, b_m) = (1, 3)$ and $(a_m, b_m) = (3, 1)$ give $\tan^2(\alpha_m) = \frac{1}{3}$ and $\tan^2(\alpha_m) = 3$ respectively. This completes the proof.

In any polynomial equation with rational coefficients, the denominators can be cleared, by multiplying the polynomial equation with a common multiple of all the denominators.

Using Theorem 2 along with the trigonometric identities (such as $\cos^2\alpha = 1/(1 + \tan^2\alpha)$, $\cos(2\alpha) = (1 - \tan^2\alpha)/(1 + \tan^2\alpha)$, $\sin(2\alpha) = 2\tan\alpha/(1 + \tan^2\alpha)$ and $\sin^2\alpha = 1 - \cos^2\alpha$), we conclude that $\cos^2(\alpha)\,, \sin^2(\alpha) \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$. From this set, we conclude that $\cos\alpha\,, \sin\alpha \in \{0, \pm\frac{1}{2}, \pm 1\}$ and $\sec\alpha\,, \csc\alpha \in \{\pm 1, \pm 2\}$. See *Box* 1 for a note on the series for arctangent function.

Box 2. At Least One of the Numbers $(\pi + e)$ or πe is Transcendental.

In general, for any two transcendental numbers T_1 and T_2 at least one of $(T_1 + T_2)$ and T_1T_2 must be transcendental. This can be seen by taking the polynomial $(x - T_1)(x - T_2) = x^2 - (T_1 + T_2)x + T_1T_2$. If $(T_1 + T_2)$ and T_1T_2 were both algebraic, then this would be a polynomial with algebraic coefficients. This would imply that the roots of the polynomial, T_1 and T_2, must be algebraic. But this is a contradiction, and thus it must be the case that at least one of the coefficients is transcendental. Choosing $T_1 = \pi$ and $T_2 = e$ proves the statement that at least one of the numbers $(\pi + e)$ or πe is transcendental.

3. Algebraic Numbers and Transcendental Numbers

In any polynomial equation with rational coefficients, the denominators can be cleared, by multiplying the polynomial equation with a common multiple of all the denominators. Then the equivalent polynomial equation thus obtained has only integer coefficients. A polynomial is said to *irreducible*, when it cannot be factored into lower degree polynomials with integer coefficients. A number is said to be *algebraic*, if it is a root of a polynomial with integer coefficients (or equivalently rational coefficients). For example, the rational numbers m/n satisfy the linear equation $nx - m = 0$. Another example is, $\sqrt{2}$, which satisfies the equation $x^2 - 2 = 0$. If a real (or complex) number is a root of an irreducible polynomial of degree n with integer coefficients, we say that, it is an algebraic number with *algebraic degree n*. The corresponding irreducible polynomial whose top coefficient is 1 is its *minimal polynomial*. Rational numbers are of *algebraic degree* 1 and $\sqrt{2}$ is of algebraic degree 2. An example of a complex algebraic number is $i = \sqrt{-1}$ and as it satisfies the equation $x^2+1 = 0$, it is of algebraic degree 2. See *Box* 2 for conditional deduction of irrationality.

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The function, $\cos(n\theta)$ can be expressed in a polynomial in $\cos(\theta) = x$ with integer coefficients. This can be done in several ways. A simple way is to use the multiple angle identity $(\cos(A + B) = \cos A \cos B - \sin A \sin B)$ and the double angle identity $(\cos(2\theta) = 2\cos^2\theta - 1)$ repeatedly. Another method is to use the de Moivre's

Box 3. Transcendental Numbers with Patterns in their Decimal Expansions.

1. Liouville Constant

$$\begin{aligned} L_b &= \sum_{n=1}^{\infty} 10^{-k!} \\ &= 10^{-1} + 10^{-2} + 10^{-6} \\ &\quad +10^{-24} + \ldots \\ &= 0.110001\ldots, \end{aligned}$$

in which the n-th digit after the decimal point is 1 if n is equal to $k!$ (k factorial) for some k and 0 otherwise.

2. Fredholm Constant

$$\begin{aligned} F &= \sum_{n=0}^{\infty} 10^{-2^n} \\ &= 10^{-1} + 10^{-2} + 10^{-4} \\ &\quad +10^{-8} + 10^{-16} + \ldots \\ &= 0.11010001\ldots, \end{aligned}$$

which also holds by replacing 10 with any algebraic number greater than 1.

3. Champernowne Constant

$$C = 0.12345678910111213\ldots$$

is the number obtained by concatenating the natural numbers and interpreting them as decimal digits to the right of a decimal point.

formula for the case $\theta = 2\pi k/n$ with k/n in the reduced form. With these conditions, $(\cos\theta + \mathrm{i}\sin\theta)^n = 1$. Expanding on the left side and equating real parts gives a polynomial equation in $\cos\theta$ and $\sin^2\theta$. The $\sin^2\theta$ is substituted by $(1 - \cos^2\theta)$. This results in an equation for $\cos(n\theta)$ in terms of a polynomial in $\cos\theta$. Both methods result in the following polynomials

$$
\begin{aligned}
T_0(x) &= 1\,, \\
T_1(x) &= x\,, \\
T_2(x) &= 2x^2 - 1\,, \\
T_3(x) &= 4x^3 - 3x\,, \\
T_4(x) &= 8x^4 - 8x^2 + 1\,, \qquad (10)
\end{aligned}
$$

where $T_n(x)$ are the Chebyshev polynomials of the *first kind*. A similar expansion exists for $\sin(n\theta) = (\sin\theta)U_{n-1}(\cos\theta)$, where $U_n(x)$ are the Chebyshev polynomials of the *second kind*. To summarize, sines and cosines of rational multiples of 360 degrees are *algebraic numbers*.

A number that is *not algebraic* is said to be a *transcendental number*, i.e., it is not a root of a polynomial equation with integer coefficients (or equivalently rational coefficients).

A number that is *not algebraic* is said to be a *transcendental number*, i.e., it is not a root of a polynomial equation with integer coefficients (or equivalently rational coefficients). The most widely known and extensively studied transcendental numbers are π, from the circle and e, the base of the natural logarithms. For any rational number, the decimal expansion has a repeating pattern. For example, as $1/7 = 0.142857\ldots$ and the string of six numbers keeps repeating. The decimal expansions of irrational numbers do not have any repeating patterns. To date, no pattern has been observed in the millions of decimal places of π and e. Some of the transcendental numbers exhibit different types of patterns, as seen in the examples in *Box* 3.

Proving a given number to be transcendental is difficult. There is a long list of open cases including, π/e, π^π, π^e, e^e e^{π^2}, ... Examples of proven classes of transcendental numbers include

1. e^a, if a is algebraic and nonzero.
2. The six trigonometric functions, $\sin(a)$, $\cos(a)$, $\tan(a)$, $\csc(a)$, $\sec(a)$ and $\cot(a)$, when a is algebraic, $a \neq 0$ and expressed in radians. Same is true for the six hyperbolic functions.
3. $\frac{1}{\pi}\tan^{-1}(r)$, if r is rational, $r \neq 0, \pm 1$ and $\tan^{-1}(r)$ is in radians.

Box 4. Hilbert Number

Hilbert number also known as the Gelfond–Schneider constant is the number $2^{\sqrt{2}}$. Both the Gelfond–Schneider constant and its square root, $\sqrt{2^{\sqrt{2}}} = \sqrt{2}^{\sqrt{2}}$ are transcendental numbers. It is interesting to note that

$$\begin{aligned}\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} &= \left(\sqrt{2}\right)^{(\sqrt{2}\sqrt{2})} \\ &= \left(\sqrt{2}\right)^{2} \\ &= 2\,.\end{aligned}$$

Thus, it is seen that a transcendental number raised to the power of an irrational number can result in a rational number!

4. The natural logarithm $\ln(a)$, if a is algebraic and $a \neq 0, 1$.

5. *Gelfond–Schneider Theorem*: a^b is transcendental, if a and b are algebraic numbers with $a \neq 0, 1$, and b irrational. If the restriction that a and b are algebraic is removed, then, the statement does not remain true in general (see *Box* 4).

6. *Baker's Theorem*: If $\alpha_1, \ldots, \alpha_m$ are non-zero algebraic numbers such that $\log\alpha_1, \ldots, \log\alpha_m$ are linearly independent over rational numbers, then $1, \log\alpha_1, \ldots, \log\alpha_m$ are linearly independent over irrational numbers. It implies the transcendence of numbers of the form ${a_1}^{b_1} \cdots {a_n}^{b_n}$, where b_i are all algebraic, irrational, and $1, b_1, \ldots b_n$ are linearly independent over the rationals, and the a_i are all algebraic and $a_i \neq 0, 1$.

7. $e^{\pi\sqrt{n}}$, where n is any natural number.

Irrational numbers can also be expressed as infinite continued fractions (see *Box* 5).

4. Deducing Irrationality Using Calculus-based Techniques

Trigonometric functions have their origins in the triangle geometry and are now found in diverse areas of mathematics. They can also be seen as solutions of differential equations. Hence, it is very natural to employ calculus-based techniques to deduce the irrationality and transcendence of trigonometric functions. Here, we shall describe a technique using the following lemma.

Trigonometric functions have their origins in the triangle geometry and are now found in diverse areas of mathematics. They can also be seen as solutions of differential equations.

Lemma. For some fixed natural number $n \geq 1$, we define

$$f(x) = \frac{1}{n!} x^n (1-x)^n , \tag{11}$$

which has the following properties

(i) The function $f(x)$ of the lemma is a polynomial of the form $f(x) = \frac{1}{n!} \sum_{i=n}^{2n} c_i x^i$, where the coefficients c_i are integers related to the binomial coefficients $\binom{n}{i} = \frac{n!}{i!(n-1)!}$ as $c_i = (-1)^i \binom{n}{i}$.

(ii) For $0 < x < 1$, we have $0 < f(x) < \frac{1}{n!}$.

(iii) The derivatives $f^{(k)}(0)$ and $f^{(k)}(1)$ are integers for all $k \geq 0$.

Proof. Parts (i) and (ii) are straightforward. From the polynomial in (i), the k-th derivative $f^{(k)}$ vanishes at $x = 0$ for $1 \leq k < n$. For $n \leq k \leq 2n$, $f^{(k)}(0) = \frac{k!}{n!} c_k$, which is an integer. Since, $f(x) = f(1-x)$ for all x, we have $f^{(k)}(x) = (-1)^k f^{(k)}(1-x)$ for all x. Consequently, $f^{(k)}(1) = (-1)^k f^{(k)}(0)$, which is an integer. Thus concluding Part (iii).

Theorem 3. *The numbers π and π^2 are irrational.*

The numbers π and π^2 are irrational.

Proof. It suffices to prove that π^2 is irrational. Let us assume that $\pi^2 = \frac{a}{b}$ for some integers, $a, b > 0$. We define a new polynomial using the function $f(x)$ of the lemma

$$\begin{aligned} F(x) \quad := \quad & b^n \left\{ \pi^{2n} f(x) - \pi^{2n-2} f^{(2)}(x) + \pi^{2n-4} f^{(4)}(x) \right. \\ & \left. - \cdots + (-1)^n f^{(2n)}(x) \right\} . \end{aligned} \tag{12}$$

Box 5. Continued Fraction Representation of π

Irrational numbers can be represented as infinite continued fractions. Initial segments provide rational approximations, and these rational numbers are called the convergents of the continued fraction. For π, we have

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}}\,.$$

No pattern has been found in this representation. The first few convergents are

$$\begin{aligned} [3] &= 3 \\ [3;7] &= \frac{22}{7} \\ [3;7,15] &= \frac{333}{106} \\ [3;7,15,1] &= \frac{355}{113} \\ [3;7,15,1,292] &= \frac{103993}{33102}\,. \end{aligned}$$

The differences of $\pi = 3.14159265358979323\ldots$ and the convergents alternate in sign.

Using property (iii) of the lemma, we conclude that $F(0)$ and $F(1)$ are integers. It is straightforward to see that $F(x)$ satisfies

$$F^{(2)}(x) + \pi^2 F(x) = b^n \pi^{2n+2} f(x) = \pi^2 a^n f(x)\,. \tag{13}$$

Then

$$\begin{aligned} &\frac{d}{dx}\left[F^{(1)}(x)\sin(\pi x) - \pi F(x)\cos(\pi x)\right] \\ &\quad = \left(F^{(2)}(x) + \pi^2 F(x)\right)\sin(\pi x) = \pi^2 a^n f(x)\sin(\pi x)\,. \end{aligned} \tag{14}$$

Now, we evaluate the integral

$$\begin{aligned} N &:= \pi \int_0^1 a^n f(x)\sin(\pi x)dx \\ &= \left[\frac{1}{\pi}F^{(1)}(x)\sin(\pi x) - F(x)\cos(\pi x)\right]\Bigg|_0^1 \\ &= F(0) + F(1)\,, \end{aligned} \tag{15}$$

which is an integer. On the other hand, from (ii) of the lemma with n sufficiently large such that $\frac{\pi a^n}{n!} < 1$, we obtain

$$0 < N = \pi \int_0^1 a^n f(x) \sin(\pi x) dx < \frac{\pi a^n}{n!} < 1 . \tag{16}$$

This is a contradiction. Hence, π^2 and π are irrational.

Using the function of the lemma (and its variants) with suitable choices of $F(x)$, the technique can be used for proving the irrationality of several classes of numbers. Examples include the trigonometric functions, e^r for any rational r and $r \neq 0$, and the hyperbolic functions. Ram Murty and Kumar Murty generalized the method, which is summarized in the following theorem.

Theorem 4. *Ram Murty–Kumar Murty Theorem: Let $G(x)$ be a non-trivial solution of the differential equation*

$$p_0 u^{(n)} + p_1 u^{(n-1)} + p_2 u^{(n-2)} + \cdots + p_n u = 0,$$

where p_i are rational numbers and $p_n \neq 0$. If $b > 0$ is such that $G(x) \geq 0$ on $[0, b]$ and $G^{(i)}(0)$, $G^{(i)}(b)$ are rational for $0 \leq i \leq n-1$, then b is irrational.

The proof makes use of the function of the lemma and an integral analogous to the one used in proving that π is irrational. We note the following applications of the powerful theorem.

(I) *The numbers π and π^2 are irrational.*
Proof. If π^2 is rational, consider $y'' + \pi^2 y = 0$ which has a solution $G(x) = \frac{1}{\pi}\sin(\pi x)$. For $b = 1$, we get a contradiction.

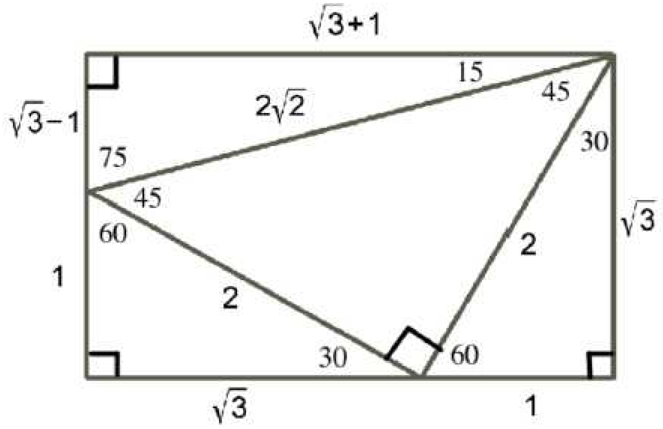
sin r and cos r are irrational for every non-zero rational r.

(II) $\sin r$ *and* $\cos r$ *are irrational for every non-zero rational r.*
Proof. If $\sin r$ is rational, consider $y'' + y = 0$ which has a solution $G(x) = \sin x$. For $b = 1$, we get a contradiction. Choosing the other solution $G(x) = \cos x$ also leads to a contradiction.

(III) *e^r is irrational for every non-zero rational r.*
Proof. If e^r is rational, consider $y' - y = 0$ which has the solution $G(x) = e^x$. For $b = 1$, we get a contradiction.

e^r is irrational for every non-zero rational r.

Box 6. Ailles Rectangle



Douglas S. Ailles, a high school teacher at Etobicoke Collegiate Institute in Etobicoke, Ontario, Canada came up with this incredibly simple method of computing the trigonometric ratios of 15° and 75°. The Ailles rectangle also gives the trigonometric ratios of 30°, 45° and 60°. Ailles published his work in 1971. A similar rectangle for other angles such as 18° and 72° has remained elusive!

(IV) $\sinh r$ *and* $\cosh r$ *are irrational for every non-zero rational* r.
Proof. If $\sinh r$ is rational, consider $y'' - y = 0$ which has a solution $G(x) = \sinh x$. For $b = 1$, we get a contradiction. Choosing the other solution $G(x) = \cosh x$ also leads to a contradiction.

5. Concluding Remarks

We approached the topic of irrational numbers using elementary trigonometry. The irrational sets of trigonometric ratios of rational angles were discussed in detail along with two major theorems and their proofs. We also covered the related topic of algebraic numbers. We obtained the expansion of $\cos(n\theta)$ as a polynomial in $\cos\theta$ with integer coefficients. This enabled us to conclude that the sines and cosines of rational multiples of 360 degrees are *al-*

gebraic numbers. We had a brief look at the transcendental numbers and noted examples of the proven classes of transcendental numbers. Using calculus-based techniques, we could prove the irrationality of π and π^2. The Ram Murty–Kumar Murty theorem based on differential equations enabled us to revisit the proof of irrationality of π and π^2. Significantly, the powerful differential equations approach enabled us to establish the irrationality of the functions $\sin r$, $\cos r$, e^r, $\sinh r$ and $\cosh r$ for every non-zero rational r. It is interesting to note that the topics of frontiers of research can be approached using some elementary trigonometry and basic calculus.

Box 6 has an interesting geometric construction, from which one can directly read the trigonometric ratios of 15° and 75°.

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Suggested Reading

[1] Eli Maor, *Trigonometric Delights*, Princeton University Press, (2013); https://doi.org/10.1515/9781400846757.

[2] Ivan Niven, Irrational numbers, *The Carus Mathematical Monographs*, No.11, The Mathematical Association of America, John Wiley and Sons, New York, 1956; https://bookstore.ams.org/cdn-1595313161227/car-11/.

[3] Ivan Niven, *Numbers: Rational and Irrational*, The Mathematical Association of America, 1961; https://www.maa.org/press/maa-reviews/numbers-rational-and-irrational.

[4] Jörg Jahnel, When co(sin) of a rational angle equal to a rational number?, arXiv:1006.2938 [math.HO], 15 June 2010; https://arxiv.org/abs/1006.2938.

[5] Bonaventura Paolillo and Giovanni Vincenzi, An elementary proof of Niven's theorem via the tangent function, *International Journal of Mathematical Education in Science and Technology*, 51, pp.1–6, 2020; https://doi.org/10.1080/0020739X.2020.1799088.

[6] M Ram Murty and V Kumar Murty, Irrational numbers arising from certain differential equations, *Canadian Mathematical Bulletin*, Vol.20, No.1, pp.117–120, 1977; https://doi.org/10.4153/CMB-1977-021-x.

[7] Michel Waldschmidt Pierre Moussa, Jean-Marc Luck and Claude Itzykson (Editors), *From Number Theory to Physics*, Springer, 1992; http://dx.doi.org/10.1007/978-3-662-02838-4.

[8] Douglas S. Ailles, Triangles and trigonometry, *Mathematics Teacher*, Vol.64, No.6, pp.562, 1971; https://doi.org/10.5951/MT.64.6.0562.

[9] Sameen Ahmed Khan, Trigonometric ratios using geometric methods, *Advances in Mathematics: Scientific Journal*, Vol.9, No.10, pp.8685–8702, 2020; https://doi.org/10.37418/amsj.9.10.94.

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