Newton's second law in action

In many cases, the nature of the force acting on a body is known. It might depend on time, position, velocity, or some combination of these, but its dependence is known from experiment.

In such cases, Newton's law becomes an *equation of motion* which we can solve. The solution allows us to predict the position of the body at any time, as long as we know its initial position and velocity. This predictive quality is the main power of Newton's law.

In this section, we will use Newton's second law in this way. We will consider the case of one-dimensional motion under a constant force, both with and without damping. This has many applications, one of which is vertical motion near earth's surface.

Motion under a constant force

When we say that the force acting on a body is constant, we mean that it does not vary over time, is the same no matter where the body is, and does not depend on the body's velocity. In other words, F is just a number. The following graph shows a case in which the constant force happens to be in the positive direction:

We are going to use Newton's law to show that the graph of the position of the body versus time, $x(t)$, is a parabola.

We begin with Newton's law, $F=ma$. Using the definition of the acceleration as the second derivative of $x(t)$, we find the following equation of motion:

$$m\frac{d^2x(t)}{dt^2} = F \quad .$$

This is a *differential equation* for the function $x(t)$. That is, it is an equation whose solution is a whole *function of time*, not just an algebraic number. It says that $x(t)$ is a function whose second derivative with respect to time is a constant, F/m. Therefore, we know what its first derivative must be; that is, we can integrate once and find

$$\frac{dx(t)}{dt} = \frac{F}{m}t + \text{constant},$$

where the constant does not depend on time. Now, the left-hand side is the velocity. So the constant must be fixed by the value of the velocity at some particular time. For simplicity, let's suppose that this time is t=0, and that the velocity at that time is v_0. Substituting t=0 into both sides, we find that the value of the constant is v_0. The velocity is therefore given by

$$v(t) = \frac{F}{m}t + v_0 \ .$$

We then integrate again, and find

$$x(t) = \frac{1}{2}\frac{F}{m}t^2 + v_0 t + \text{another constant.}$$

The new constant is fixed by the value of the position at another particular time, which we may again take to be t =0. If the position then is x_0, then the value of the new constant is x_0. The final answer for the position as a function of time is thus given by

$$\boxed{x(t) = \frac{F}{2m}t^2 + v_0 t + x_0} \ .$$

This formula allows us to find the position at any time, as long as we know the values of the initial position and velocity. The reason we have to know *two* quantities is because Newton's law gives rise to a *second-order* differential equation. That is, the highest derivative which appears is the second derivative.

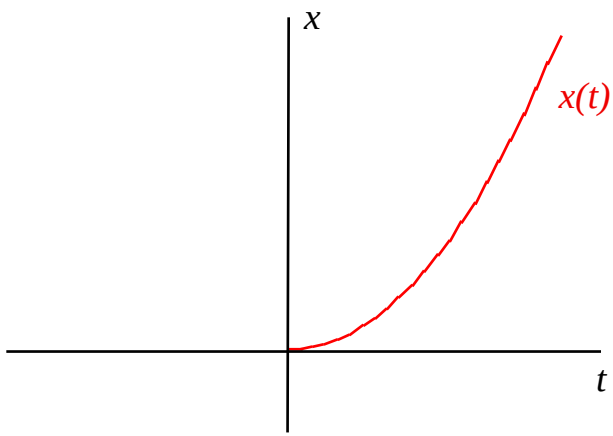
Let's try to picture such a motion physically. We will do this by considering some special cases.

Special case: zero initial velocity, positive force

Suppose the body is initially at rest at x_0=0, and the force acts in the positive x direction. Then we know what will happen - the body will just speed up in the direction of the force. The position as a function of time is given by

$$x(t) = \frac{F}{2m}t^2 \ .$$

This is (one half of) a parabola pointing upwards:

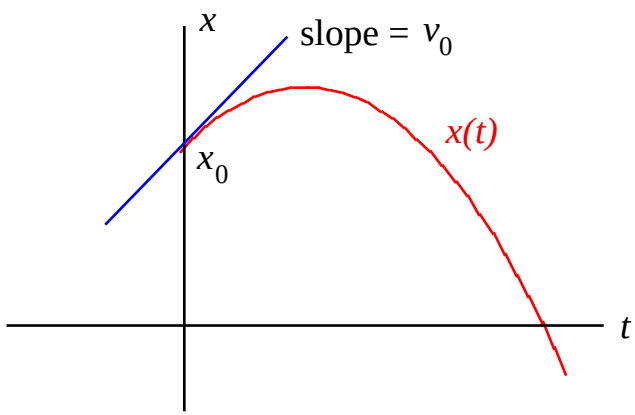


The slope at t=0 is zero, corresponding to zero initial velocity.

Special case: positive v_0, negative force

Suppose now that the initial velocity is positive, but the force is negative. That means that the body will slow down, eventually stop, and then speed up in the opposite direction. Since the force is constant, the body continues to speed up indefinitely (until Newtonian mechanics itself breaks down - a topic for later discussion).

Here is a plot of $x(t)$ in the case just described:



Because we have the solution for $x(t)$, we may answer virtually any question we like about the properties of the motion. For example, suppose we want to know the maximum value of x reached by the body, and at what time it is reached. This may be handled by standard methods of calculus, but we will instead do it here using the methods of analytic geometry. We complete the square in the solution for $x(t)$:

$$x(t)=\frac{F}{2m}\left(t+\frac{mv_0}{F}\right)^2+x_0-\frac{mv_0^2}{2F}$$

This makes it clear that the path in the t-x plane is a parabola. The parabola opens downward in our case, because the factor $F/2m$ is negative. The maximum value of x occurs when the first term is zero, i.e. when

$$t=-\frac{mv_0}{F} \quad ,$$

which is positive). The maximum value of x is

$$x_{\max}=x_0-\frac{mv_0^2}{2F} \quad ,$$

which is greater than x_0.

Here's a MAPLE input line which solves the present differential equation symbolically:

```
soln:=op(2,dsolve({m*diff(x(t),t$2)=F,
x(0)=x[0],D(x)(0)=v[0]},x(t)));
```

The next line plots a solution:

```
plot(subs({m=1,F=-1,x[0]=1,v[0]=1},
soln),t=0..3,0..2);
```

(Here's how to copy these lines from the present

document and paste them into MAPLE.) You can easily change the values of the parameters before executing the input. If you wish, you can also plot the velocity by changing `soln` to `diff(soln,t)` in the above line.

Application: vertical motion near earth

A very important physical situation to which the above applies directly is the case of a body moving vertically near the surface of the earth.

Although we will not study the gravitational interaction until later, you may be familiar with the relevant fact:

near the surface of the earth, *all* bodies (whatever their inertial mass) have an acceleration of approximately 9.81 m s^{-2} in the downward direction.

This is true as long as the effects of air resistance are negligible. This value is given a special symbol, g, and is called the

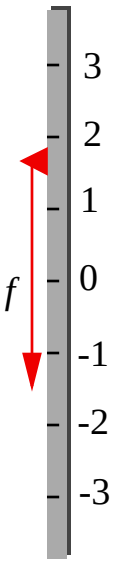
gravitational acceleration at earth's surface:

$$g \approx 9.81 \text{ m s}^{-2} .$$

Let's orient our coordinates vertically, with positive values pointing upwards. Then the force due to gravity is

$$F = -mg$$

The minus sign indicates that the force is directed downwards. (If we had oriented the coordinates with positive values pointing downwards, the minus sign would not be present in the equation for F.)



This force is constant - does not depend on time, position, or velocity. We can therefore take over the above formulas directly, with $-mg$ substituted for F . In particular, we find that

• a body fired upwards with velocity v_0 reaches a maximum height of

$$\boxed{\frac{v_0^2}{2g}}$$

above its starting height, at time

$$\boxed{\frac{v_0}{g}} \quad .$$

The path in the *t-x* plane is given by the parabola shown two figures ago. It is important to realize that the path *in space* is a straight line, not a parabola. We are considering vertical motion *only*, at present. (Later, when we come to consider motion in more than one dimension, we will find that a projectile can move on a parabolic path *in space*. That parabolic path is *not the same* as the present one in the *t-x* plane - don't get them confused!)

The resources for this section contain a movie showing this vertical motion along with graphs of position, velocity and acceleration. For comparison, there is also a movie showing the same quantities for a body dropped from rest.

Resistive forces

In many physical cases, there is some resistance to motion. For example, a body could be sliding along a track with friction present. Or, a body could be moving vertically near the earth, with air resistance.

In many cases, the force resisting the motion is proportional to the *velocity* of the body. Mathematically, this is written

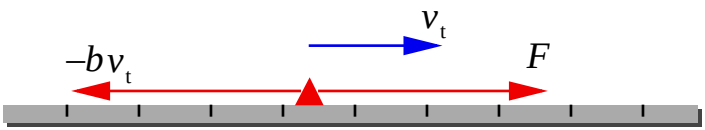
$$\boxed{F_{\text{res}} = -bv} \quad .$$

The quantity b is a positive constant, whose value depends on the properties of the material providing the resistance. It is not a fundamental constant of nature. The minus sign indicates that the force resists the motion, so is directed *opposite* to the velocity.

We would like to illustrate the procedure of solving Newton's law when such a force is involved. We will suppose that the total force on the body is the sum of the constant force F that we have just been considering, and the resistive force F_{res}:

$$F_{\text{net}} = F + F_{\text{res}} \quad .$$

It is always a good idea to use physical intuition to get an idea of the nature of the solution, before beginning the mathematics. In the present case, it is easy to see one aspect of the solution. As time goes on, the external constant force will just balance the resistive force, giving zero net force. The body will then move with a constant velocity called the *terminal velocity* v_t.



The above figure shows the equal and opposite forces in red. The net force is

$$F_{\text{net}} = F - bv_t = 0 \ ,$$

which gives for the terminal velocity

$$v_t = \frac{F}{b} \ .$$

Let's solve the equation of motion and see how this is reflected in the solution. Newton's law reads

$$m\frac{dv(t)}{dt} = F - bv(t),$$

which we have written entirely in terms of the velocity $v(t)$ and its first derivative.

How to solve this? Well, if F were zero, we would have $dv/dt=-(b/m)v$, which has its solution some constant times $\exp(-bt/m)$. By inspection, we find that we can account for nonzero F by simply adding a constant, F/b. That is,

$$v(t) = \frac{F}{b} + \text{constant} \times \exp\left(-\frac{b}{m}t\right).$$

The value of the constant must be related to the value of v at some particular time. For convenience, let's say that at $t=0$ the value of v is v_0. Then the constant is v_0-F/b. The velocity is thus given by

$$\boxed{v(t) = \frac{F}{b} + \left(v_0 - \frac{F}{b}\right)\exp\left(-\frac{b}{m}t\right)} \ .$$

As t becomes large, the second term vanishes and $v(t)$ approaches F/b, as we know it should. Notice that the second term never actually becomes zero at any finite time - it just gets closer and closer.

Let's now see how the introduction of the resistive force changes the results we found earlier in the case where the initial velocity is positive and the force is negative.

1) At what time t_{max} does the position reach its maximum value? The answer is the time at which the velocity is zero. Solving, we find that this is given by

$$t_{\text{max}} = \frac{m}{b}\ln\left(1 - \frac{bv_0}{F}\right).$$

It is positive because bv_0/F is negative.

2) What is the maximum value x_{max} of the position? To answer this, we need to solve for

$x(t)$. We integrate our expression for $v(t)$ once, obtaining

$$x(t) = \frac{F}{b}t - \frac{m}{b}\left(v_0 - \frac{F}{b}\right)\exp\left(-\frac{b}{m}t\right) + \text{constant}\ .$$

Let's say that $x = x_0$ at $t=0$. Substituting into the above expression yields

$$\text{constant} = x_0 + \frac{m}{b}\left(v_0 - \frac{F}{b}\right).$$

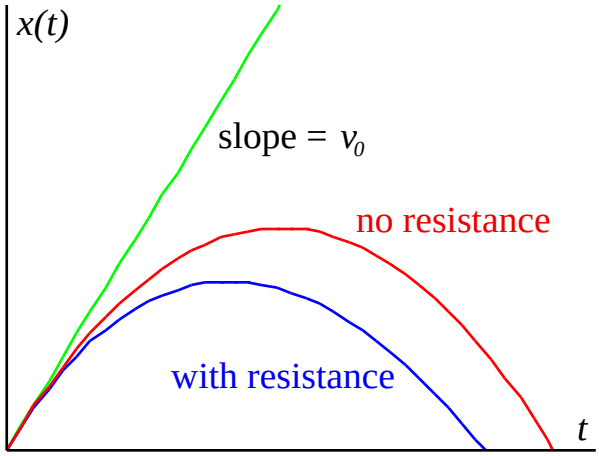
Hence, the full solution for $x(t)$ is

$$\boxed{x(t) = x_0 + \frac{F}{b}t + \frac{m}{b}\left(v_0 - \frac{F}{b}\right)\left(1 - \exp\left(-\frac{b}{m}t\right)\right)}\ .$$

To answer our question, we substitute $t_{\max}$ into our expression for $x(t)$, obtaining

$$x_{\max} = x_0 + \frac{mv_0}{b} + \frac{Fm}{b^2}\ln\left(1 - \frac{bv_0}{F}\right).$$

Here is a plot of $x(t)$ showing the effect of resistance; a plot with the same initial conditions but no resistance is shown in red for comparison. We see that the curve with resistance deviates noticeably from the ideal parabolic form.



Here is a MAPLE input line which solves the present differential equation symbolically:

```
soln:=op(2,dsolve({m*diff(x(t),t$2)=F-
b*diff(x(t),t),x(0)=x[0],D(x)(0)=v[0]},
x(t)));
```

The next line plots a solution:

```
plot(subs({m=1,F=-1,b=1,x[0]=0,v[0]=1},
soln),t=0..2);
```

(Here's how to copy these lines from the present document and paste them into MAPLE.) You can easily change the values of the parameters before executing. If you wish, you can also plot the velocity by changing `soln` to `diff(soln,t)` in the above line.

Checking our answers:

In the next section, we will consider a couple of ways to check the answers we obtained in this

section. In particular, we will see how to make sure that the units are correct (*dimensional analysis*), and also how to make sure that the case of zero damping is recovered when we set the damping to zero in the more general case.