

Numerical Integration of Newton's Second Law of Motion

One of the biggest things we've done this semester is develop Newton's Second Law of Motion: $\vec{F} = m\vec{a}$. One of the reason's this is important is operational: using Newton's laws of motion, it becomes possible to determine the position of a given object at any time (given it's position and velocity at some initial time). (A more trivial reason for the importance is that Newton's mechanics denies the existence of free will – but I digress...). Some of the earlier successes of Newton's mechanics were descriptions of the planetary motion that people had observed for centuries before. Other more mundane applications are motion of projectiles and other objects here on Earth. It is the purpose of this lab to develop the means to analyze the projectile motion of an object subject to a drag force that varies as the square of the velocity (as well as the force of gravity, of course), and to explore the behavior of such an object.

Theoretical analysis

First, consider the task of finding position as a function of time. Newton's Second tells us that:

$$\vec{F}(\vec{r}, \vec{v}, t) = m\vec{a} = m \frac{d\vec{v}(t)}{dt} = m \frac{d^2\vec{r}(t)}{dt^2} \quad (1)$$

That is, we can find position as a function of time if we know the forces involved and can solve a differential equation (an equation with derivatives all in it). Notice that the force might depend on position and time (and even velocity, the rate of change of position). In this case, the problem graduates from being merely Hard to being Really Hard. Because in order to write down the force you need to know the position and velocity – but to get the position and velocity, I need to write down the force and solve the derivatives!

Catch-22. Remember that the concept of “energy” was invented to get around some of this. In other cases, it is possible to separate the various dependences (for example, if the drag force depends only upon the velocity to the first power – called “Stokes’ drag” – it is possible to separate the variables and solve the problem). But the number of problems you can solve in this way is actually pretty small (it was, to a certain extent, lucky for Newton that the important problems were problems it is possible to solve). In the vast majority of cases, it is necessary to solve this problem numerically. We discuss what “numerically” means below, but in short it involves doing LOTS and LOTS of simple calculations. It is possible to do this by hand, but it only became common with the advent of computers (and particularly computers that can do lots of calculations really fast). So how does that help?

Imagine that I know the position – $\mathbf{r}(t_n)$ – and velocity – $\mathbf{v}(t_n)$ – of an object at one particular instant of time, t_n . (Never mind how – we'll get to that in a moment).

Newton's Second tells me that:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{\vec{v}(t_n + dt) - \vec{v}(t_n)}{dt} = \frac{\vec{F}(\vec{r}_n, \vec{v}_n, t_n)}{m} \quad (2)$$

So we see how to find the velocity at some instant later:

$$\vec{v}(t_n + dt) = \vec{v}(t_n) + \frac{\vec{F}(\vec{r}_n, \vec{v}_n, t_n)}{m} dt \quad (3)$$

And the position can be found simply from:

$$\begin{aligned} \vec{r}(t_n + dt) &= \vec{r}(t_n) + \vec{v}_{AVG} dt = \vec{r}(t_n) + \frac{\vec{v}(t_n + dt) + \vec{v}(t_n)}{2} dt \\ &= \vec{r}(t_n) + \vec{v}(t_n) dt + \frac{1}{2} \frac{\vec{F}(\vec{r}_n, \vec{v}_n, t_n)}{m} (dt)^2 \end{aligned} \quad (4)$$

Hopefully, these equations will look familiar: they are the equations we developed for the special case of constant acceleration. And that is effectively what we are doing: assuming that the acceleration does not change. This is exactly true for an infinitely short time period, and is APPROXIMATELY true for a pretty short time period. Most of the hard work in numerical calculations of the type we're undertaking is figuring out when the time periods are short enough to give you an approximate answer close enough to the right answer.

Note that all the equations given so far are vector equations. That means that they are really three equations in one (one for each dimension – x, y, and z). For the problems we will consider, the problems are actually two dimensional (the projectile objects will stay in the plane containing their initial velocity – we will not prove that here, but you SHOULD). So our problems will really be 2 dimensional, which makes life easier. The two equations of motion can be developed in the following way. The acceleration is obtained from the force in each direction:

$$\begin{aligned} a_x(t_n) &= \frac{F_x(\vec{r}_n, \vec{v}_n, t_n)}{m} \\ a_y(t_n) &= \frac{F_y(\vec{r}_n, \vec{v}_n, t_n)}{m} \end{aligned} \quad (5)$$

The velocity and position as indicated above:

$$\begin{aligned} v_x(t_n + dt) &= v_x(t_n) + a_x(t_n) dt \\ v_y(t_n + dt) &= v_y(t_n) + a_y(t_n) dt \end{aligned} \quad (6)$$

$$\begin{aligned} x(t_n + dt) &= x(t_n) + v_x(t_n) dt + \frac{1}{2} a_x(t_n) (dt)^2 \\ y(t_n + dt) &= y(t_n) + v_y(t_n) dt + \frac{1}{2} a_y(t_n) (dt)^2 \end{aligned} \quad (7)$$

The last step is to realize that if we have the initial values (position and velocity at t=0) we can then step through time and generate the entire trajectory of the particle.

Example: Simple projectile motion

A simple case is that of a projectile subject only to gravity (meaning $a_x = 0$, $a_y = -9.8 \text{ m/s}^2$). Note that this is a problem we don't need to calculate numerically. We discuss it here in order to clarify the method and to allow us to assess its accuracy. We want to calculate the trajectory of a particle for a given initial conditions. We start the projectile at the origin ($x=0$, $y=0$), with an initial speed of 10 m/s at an initial angle of 30 degrees (meaning $v_x(0) = 8.66 \text{ m/s}$, $v_y(0) = 5 \text{ m/s}$). We choose a time step of 0.1 second. Then at the next time step ($t=0.1 \text{ s}$), $v_x(0.1) = 8.66 \text{ m/s}$ and $v_y(0.1) = 5 \text{ m/s} - 9.8 \text{ m/s}^2 (0.1 \text{ s})$

= 4.02 m/s. The position is $x(0.1) = 0 \text{ m} + 8.66 \text{ m/s} (0.1 \text{ s}) = 0.866 \text{ m}$ and $y(0.1) = 0 \text{ m} + (5 \text{ m/s})(0.1 \text{ s}) + \frac{1}{2} (-9.8 \text{ m/s}^2)(0.1 \text{ s})^2 = 0.451 \text{ m}$. Knowing the values for $t=0.1 \text{ s}$ allows us to calculate the position at $t=0.2 \text{ sec}$: $v_x(0.2) = 8.66 \text{ m/s}$ and $v_y(0.2) = 4.02 \text{ m/s} - 9.8 \text{ m/s}^2 (0.1 \text{ s}) = 3.04 \text{ m/s}$. The position is $x(0.2) = 0.866 \text{ m} + 8.66 \text{ m/s} (0.1 \text{ s}) = 1.732 \text{ m}$ and $y(0.2) = 0.451 \text{ m} + (4.02 \text{ m/s})(0.1 \text{ s}) + \frac{1}{2} (-9.8 \text{ m/s}^2)(0.1 \text{ s})^2 = 0.804 \text{ m}$.

This is already tedious and there are still about 10 more iterations to go – that’s why people do this with computers rather than by hand (with a calculator). But if we carry this out, we get a table of values that looks like:

| time(sec) | x position | y position | vx | vy | ax | ay |
|-----------|------------|------------|------|-------|------|-------|
| 0.00 | 0.00 | 0.00 | 8.66 | 5.00 | 0.00 | -9.80 |
| 0.10 | 0.87 | 0.40 | 8.66 | 4.02 | 0.00 | -9.80 |
| 0.20 | 1.73 | 0.71 | 8.66 | 3.04 | 0.00 | -9.80 |
| 0.30 | 2.60 | 0.91 | 8.66 | 2.06 | 0.00 | -9.80 |
| 0.40 | 3.46 | 1.02 | 8.66 | 1.08 | 0.00 | -9.80 |
| 0.50 | 4.33 | 1.03 | 8.66 | 0.10 | 0.00 | -9.80 |
| 0.60 | 5.20 | 0.94 | 8.66 | -0.88 | 0.00 | -9.80 |
| 0.70 | 6.06 | 0.76 | 8.66 | -1.86 | 0.00 | -9.80 |
| 0.80 | 6.93 | 0.47 | 8.66 | -2.84 | 0.00 | -9.80 |
| 0.90 | 7.79 | 0.09 | 8.66 | -3.82 | 0.00 | -9.80 |
| 1.00 | 8.66 | -0.39 | 8.66 | -4.80 | 0.00 | -9.80 |
| 1.10 | 9.53 | -0.97 | 8.66 | -5.78 | 0.00 | -9.80 |

I stopped here because the y-position is becoming negative (meaning that the object has already hit the floor). And we can look at the results and determine that the range is between 7.8 m and 8.6 m (probably closer to 7.8 m), the maximum height is about 1.03 m, the time to the maximum height is between 0.4 sec and 0.5 sec (probably about 0.45 sec), and the time back to the ground is between 0.9 sec and 1.0 sec (probably closer to 0.9 sec). And we can check these results because we know how to calculate all these quantities for simple projectile motion: $R = (v_0^2/g) \sin 2\Theta = 8.837 \text{ m}$, $y_{\text{MAX}} = (v_0 \sin \Theta)^2/2g = 1.276 \text{ m}$, time for y_{MAX} is $t = (v_0 \sin \Theta)/g = 0.510 \text{ sec}$ and time for R is $t = (2v_0 \sin \Theta)/g = 1.020 \text{ sec}$.

Notice that all the results from the numerical calculation are close (but no cigar). And they are all a little too small. This is because our approximation (that acceleration is constant) fails, at least a little. For example, we calculate the position at the end of 0.5 second as $x = 4.33 \text{ m}$ and $y = 1.03 \text{ m}$. But an exact calculation gives a result of $x = 4.33 \text{ m}$ and $y = 1.275 \text{ m}$. It’s complicated to explain, but there are two reasons our numerical results are different from analytical results: our time steps are too big and our method of calculating the position is too simple-minded. There is an entire branch of math and physics devoted just to the study of more intelligent ways to solve differential equations. But fortunately for us, we don’t have to work smarter, we can make the computer work harder. If we do the same calculation as described above, but using steps of 0.01 sec, we get: $R = 8.747 \text{ m}$, $y_{\text{MAX}} = 1.2505 \text{ m}$, time for y_{MAX} is $t = 0.51 \text{ sec}$ and time for R is $t = 1.01 \text{ sec}$. Note that the results are much closer this go round (and we can get closer still by shortening the step size more). You can find the spreadsheet (in Excel format) that I used in these calculations and screw around with it some. For example, you could calculate R while varying the angle, plot R versus Θ and conclude that the maximum range is for $\Theta = 45^\circ$. (With some effort you might even be able to figure out

that $R \sim \sin(2\Theta)$, but it's harder than with the analytical technique). But the REAL point of this lab is to do something you CAN'T do analytically: a projectile undergoing quadratic drag.

Projectile with Quadratic Drag

A *real* object flying through the air feels a drag force that depends on the speed of the object. For “low” speeds, that force is linear in the speed (i.e., $F \sim v$), while for “high” speeds, the force is quadratic (i.e., $F \sim v^2$). For a baseball, for example, “low” speeds are speeds much, much less than about 1 mm/sec (that's millimeters...). High speeds are much, much greater than 1 mm/sec. So a fastball is DEFINITELY in the quadratic regime. (Aside: This depends only on the SIZE of the object, so a cannon ball is the same as a baseball for these purposes – anybody who wants to know how this is calculated should sing out).

Unfortunately for us, it is possible to solve the problem of linear drag analytically (and also quadratic drag in one dimension – we did this in class), but NOT quadratic drag in two dimensions. So we'll do it numerically. Your task is to implement the solution in whatever calculating program you want (including Fortran, if that's your bag). I'll sketch out the solution here.

First, the forces are:

$$\vec{F}_X(\vec{r}_n, \vec{v}_n, t_n) = mg(-\hat{j}) + \frac{1}{2} \rho A v^2 (-\hat{v}) \quad (7)$$

Note that I've defined down as negative (right?) and the drag force points in a direction opposite the velocity (right?). There's no explicit time dependence (good!) and the only space dependence is trivial. The acceleration in each direction is then:

$$a_x(t_n) = -\frac{\rho A v^2}{2m} \cos \theta \quad (8)$$

$$a_y(t_n) = -g - \frac{\rho A v^2}{2m} \sin \theta$$

But note that we must be VERY CAREFUL that the directions are correct – we have written the drag force along y as negative, but this is only true if the object is going up. On the way down, the drag force is actually UP (against the velocity). If we write the trig functions as $\cos \Theta = v_x/v$ and $\sin \Theta = v_y/v$, then the problem will take care of itself (you should prove this):

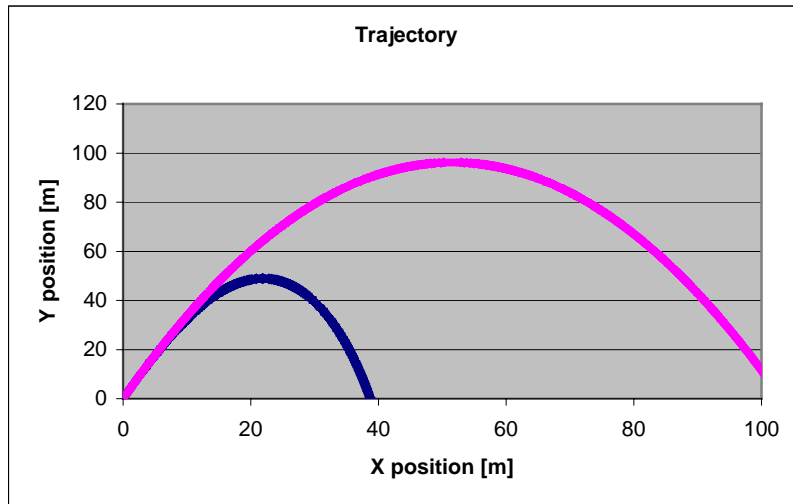
$$a_x(t_n) = -\frac{\rho A v^2}{2m} \frac{v_x}{v} = -\frac{\rho A v v_x}{2m} \quad (9)$$

$$a_y(t_n) = -g - \frac{\rho A v^2}{2m} \frac{v_y}{v} = -g - \frac{\rho A v v_y}{2m}$$

The velocity and position are then calculated in precisely the same way as we did for constant acceleration. But note that now we don't have the benefit of knowing the answer to compare to our calculation. So how do we know our answer is right? We don't. All we can do is work hard and try to be careful and hope and if possible, compare our answer to some experiment. (That's why theoretical physicists are always desperate for experimental data to compare to their calculations – if not, they're really mathematicians). Here's a graph of the results I had for the following conditions:

| | | | | |
|---------------------------------|----------|--|-------------|-------|
| gravity plus quadratic air drag | | | mass= | 0.145 |
| v0 = | 45 | | radius= | 0.03 |
| theta0= | 75 | | air density | 1.2 |
| k= | 0.011701 | | | |

(the k thing is just the constant parts of the drag force ($A/2m$) to make the calculation slightly easier to code) The graph looks like:



The blue line is the trajectory for the object with air drag and the pink line (excuse me – magenta) is without air drag.

Conclusion

The tough part is getting the calculations to work. Once it does, it gets kind of fun (if your mind works that way) because you can then start playing – for example, try varying the radius without varying the mass (you'll see why you can throw a baseball much farther than a beach ball, even though they weigh about the same). Also, you can vary the angle without changing anything else – you'll see that with air drag, the maximum range does NOT occur at a launch angle of 45° . (For the conditions I show you above, the maximum range occurs for a launch angle of 38°). You could even construct a plot of the range as a function of launch angle (that would be a good idea for a report). You should discuss in a report (if you do one) why defining the sin and cosine the way we did is correct and gives the right direction for the acceleration.

Finally, suppose some nitwit in New Orleans decides it would be a great idea to fire off his pistol in the air on New Years Eve. How high would it go? How fast would it be going when it landed? How far would it go to the side (if he shoots in the Marigny, will he kill somebody in the Quarter? How about Uptown?)?

Finally (really this time), if you drop a penny off the Empire State Building, would it be going fast enough to kill somebody if it hit them? How about if you spit?