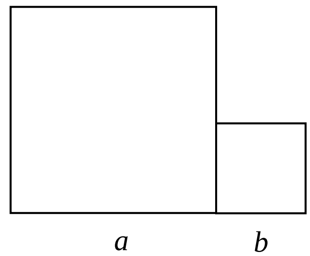
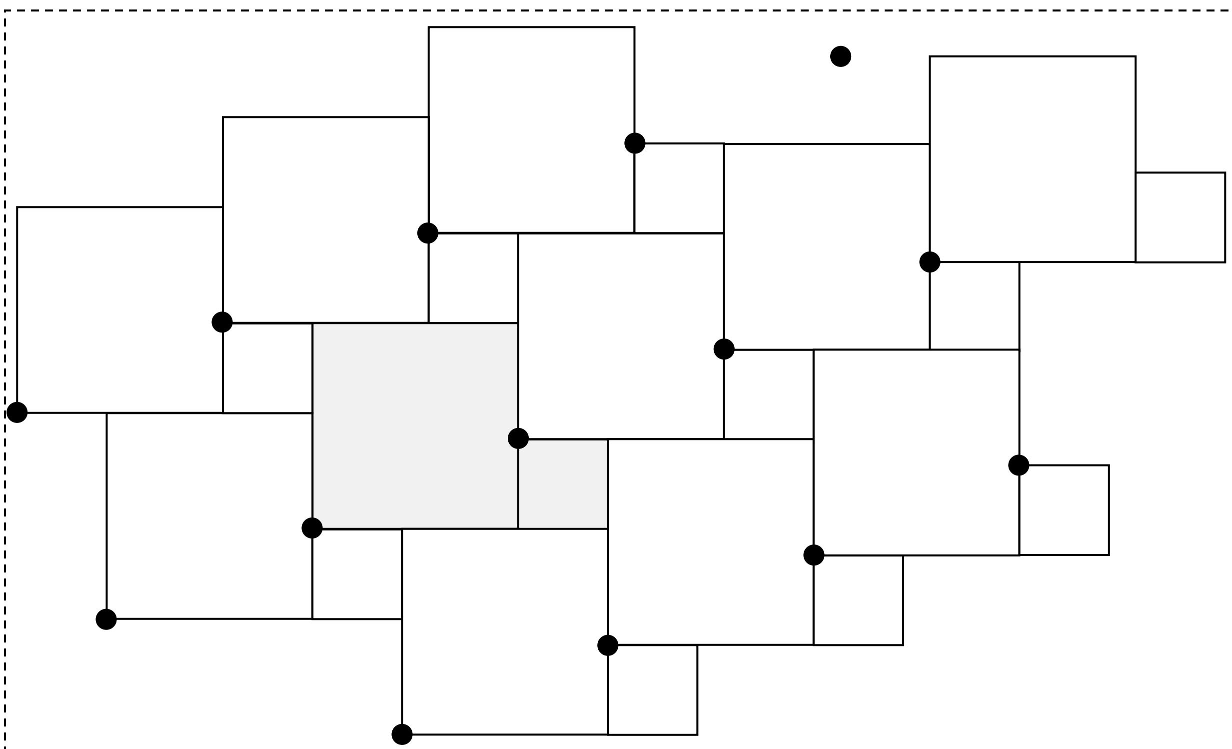
Proof of Pythagorean Theorem Using Lattices

Adapted from Roger Penrose [1].

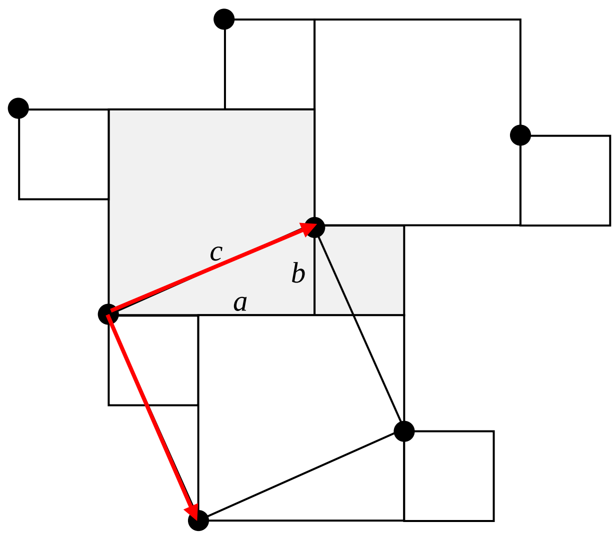
Let a and b be real numbers such that (without loss of generality) $b \leq a$. Place two squares of sides a and b respectively side by side to form a tile, as follows.



We can tessellate the plane with this tile by arranging the tiles as shown below, where the pattern continues indefinitely in all directions. We can associate a lattice point with the lower left corner of each tile, as shown in the figure. The tile is a thus a unit cell of this lattice, and has area $a^2 + b^2$. Thus, the area of any unit cell of this lattice must be $a^2 + b^2$, which is the determinant of the lattice.



The figure below shows a portion of the previous figure with a suitable set of basis vectors for the lattice (in red). We see that the two basis vectors are each the hypotenuse of two congruent right-angled triangles with sides *a* and *b*. We also can see from these two triangles that the angle between the basis vectors is 90°. Thus the basis vectors form two sides of a square which is clearly a unit cell of the lattice. If we call its side *c*, then the area of the unit cell is c^2.



We have thus deduced that $c^2 = a^2 + b^2$.

Note that a sampling matrix for the lattice is $\mathbf{V}_\Lambda = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ and $d(\Lambda) = a^2 + b^2$.

[1] R. Penrose, *The Road to Reality: A Complete Guide to the Laws of the Universe*, New York: Knopf, 2005, pp. 25-28.