AP® Calculus AB
2003 Scoring Guidelines
Form B

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Question 1

Let f be the function given by $f(x) = 4x^2 - x^3$, and let ℓ be the line $y = 18 - 3x$, where ℓ is tangent to the graph of f. Let R be the region bounded by the graph of f and the x-axis, and let S be the region bounded by the graph of f, the line ℓ, and the x-axis, as shown above.
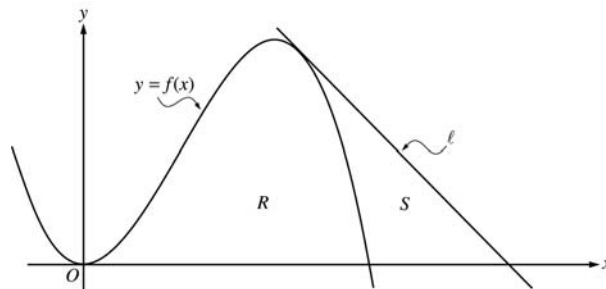
(a) Show that ℓ is tangent to the graph of $y = f(x)$ at the point $x = 3$.

(b) Find the area of S.

(c) Find the volume of the solid generated when R is revolved about the x-axis.

(a) $f'(x) = 8x - 3x^2$; $f'(3) = 24 - 27 = -3$

$f(3) = 36 - 27 = 9$

Tangent line at $x = 3$ is

$y = -3(x - 3) + 9 = -3x + 18$,

which is the equation of line ℓ.

$$2 : \begin{cases} 1 : \text{finds } f'(3) \text{ and } f(3) \\ 1 : \begin{cases} \text{finds equation of tangent line} \\ \qquad\qquad \text{or} \\ \text{shows } (3,9) \text{ is on both the} \\ \text{graph of } f \text{ and line } \ell \end{cases} \end{cases}$$

(b) $f(x) = 0$ at $x = 4$

The line intersects the x-axis at $x = 6$.

$\text{Area} = \dfrac{1}{2}(3)(9) - \displaystyle\int_3^4 \left(4x^2 - x^3\right) dx$

$= 7.916$ or 7.917

OR

$\text{Area} = \displaystyle\int_3^4 \left((18 - 3x) - \left(4x^2 - x^3\right)\right) dx$

$\qquad + \dfrac{1}{2}(2)(18 - 12)$

$= 7.916$ or 7.917

$$4 : \begin{cases} 2 : \text{integral for non-triangular region} \\ \qquad 1 : \text{limits} \\ \qquad 1 : \text{integrand} \\ 1 : \text{area of triangular region} \\ 1 : \text{answer} \end{cases}$$

(c) $\text{Volume} = \pi \displaystyle\int_0^4 \left(4x^2 - x^3\right)^2 dx$

$= 156.038\,\pi$ or 490.208

$$3 : \begin{cases} 1 : \text{limits and constant} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$$

Question 2

A tank contains 125 gallons of heating oil at time $t = 0$. During the time interval $0 \le t \le 12$ hours, heating oil is pumped into the tank at the rate

$$H(t) = 2 + \frac{10}{(1 + \ln(t+1))} \text{ gallons per hour.}$$

During the same time interval, heating oil is removed from the tank at the rate

$$R(t) = 12 \sin\left(\frac{t^2}{47}\right) \text{ gallons per hour.}$$

(a) How many gallons of heating oil are pumped into the tank during the time interval $0 \le t \le 12$ hours?

(b) Is the level of heating oil in the tank rising or falling at time $t = 6$ hours? Give a reason for your answer.

(c) How many gallons of heating oil are in the tank at time $t = 12$ hours?

(d) At what time t, for $0 \le t \le 12$, is the volume of heating oil in the tank the least? Show the analysis that leads to your conclusion.

(a) $\displaystyle\int_0^{12} H(t)\,dt = 70.570$ or 70.571

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $H(6) - R(6) = -2.924$,
so the level of heating oil is falling at $t = 6$.

$1 :$ answer with reason

(c) $125 + \displaystyle\int_0^{12}(H(t) - R(t))\,dt = 122.025$ or 122.026

$3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(d) The absolute minimum occurs at a critical point or an endpoint.
$H(t) - R(t) = 0$ when $t = 4.790$ and $t = 11.318$.

The volume increases until $t = 4.790$, then decreases until $t = 11.318$, then increases, so the absolute minimum will be at $t = 0$ or at $t = 11.318$.

$125 + \displaystyle\int_0^{11.318}(H(t) - R(t))\,dt = 120.738$

Since the volume is 125 at $t = 0$, the volume is least at $t = 11.318$.

$3 : \begin{cases} 1 : \text{sets } H(t) - R(t) = 0 \\ 1 : \text{volume is least at} \\ \quad\ t = 11.318 \\ 1 : \text{analysis for absolute} \\ \quad\ \text{minimum} \end{cases}$

Question 3

A blood vessel is 360 millimeters (mm) long with circular cross sections of varying diameter. The table above gives the measurements of the diameter of the blood vessel at selected points

Distance x (mm)	0	60	120	180	240	300	360
Diameter $B(x)$ (mm)	24	30	28	30	26	24	26

along the length of the blood vessel, where x represents the distance from one end of the blood vessel and $B(x)$ is a twice-differentiable function that represents the diameter at that point.

(a) Write an integral expression in terms of $B(x)$ that represents the average radius, in mm, of the blood vessel between $x = 0$ and $x = 360$.

(b) Approximate the value of your answer from part (a) using the data from the table and a midpoint Riemann sum with three subintervals of equal length. Show the computations that lead to your answer.

(c) Using correct units, explain the meaning of $\pi \int_{125}^{275} \left(\dfrac{B(x)}{2} \right)^2 dx$ in terms of the blood vessel.

(d) Explain why there must be at least one value x, for $0 < x < 360$, such that $B''(x) = 0$.

(a) $\dfrac{1}{360} \displaystyle\int_0^{360} \dfrac{B(x)}{2} \, dx$

$2 : \begin{cases} 1 : \text{limits and constant} \\ 1 : \text{integrand} \end{cases}$

(b) $\dfrac{1}{360}\left[120\left(\dfrac{B(60)}{2} + \dfrac{B(180)}{2} + \dfrac{B(300)}{2} \right) \right] =$

$\dfrac{1}{360}[60(30 + 30 + 24)] = 14$

$2 : \begin{cases} 1 : B(60) + B(180) + B(300) \\ 1 : \text{answer} \end{cases}$

(c) $\dfrac{B(x)}{2}$ is the radius, so $\pi\left(\dfrac{B(x)}{2} \right)^2$ is the area of the cross section at x. The expression is the volume in mm^3 of the blood vessel between 125 mm and 275 mm from the end of the vessel.

$2 : \begin{cases} 1 : \text{volume in mm}^3 \\ 1 : \text{between } x = 125 \text{ and} \\ \quad x = 275 \end{cases}$

(d) By the MVT, $B'(c_1) = 0$ for some c_1 in $(60, 180)$ and $B'(c_2) = 0$ for some c_2 in $(240, 360)$. The MVT applied to $B'(x)$ shows that $B''(x) = 0$ for some x in the interval (c_1, c_2).

$3 : \begin{cases} 2 : \text{explains why there are two} \\ \quad \text{values of } x \text{ where } B'(x) \text{ has} \\ \quad \text{the same value} \\ 1 : \text{explains why that means} \\ \quad B''(x) = 0 \text{ for } 0 < x < 360 \end{cases}$

Note: max 1/3 if only explains why
$B'(x) = 0$ at some x in $(0, 360)$.

Question 4

A particle moves along the x-axis with velocity at time $t \geq 0$ given by $v(t) = -1 + e^{1-t}$.

(a) Find the acceleration of the particle at time $t = 3$.

(b) Is the speed of the particle increasing at time $t = 3$? Give a reason for your answer.

(c) Find all values of t at which the particle changes direction. Justify your answer.

(d) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 3$.

(a) $a(t) = v'(t) = -e^{1-t}$

$a(3) = -e^{-2}$

$2 : \begin{cases} 1 : v'(t) \\ 1 : a(3) \end{cases}$

(b) $a(3) < 0$

$v(3) = -1 + e^{-2} < 0$

Speed is increasing since $v(3) < 0$ and $a(3) < 0$.

$1 :$ answer with reason

(c) $v(t) = 0$ when $1 = e^{1-t}$, so $t = 1$.

$v(t) > 0$ for $t < 1$ and $v(t) < 0$ for $t > 1$.

Therefore, the particle changes direction at $t = 1$.

$2 : \begin{cases} 1 : \text{solves } v(t) = 0 \text{ to} \\ \quad \text{get } t = 1 \\ 1 : \text{justifies change in} \\ \quad \text{direction at } t = 1 \end{cases}$

(d) Distance $= \int_0^3 |v(t)| \, dt$

$= \int_0^1 \left(-1 + e^{1-t} \right) dt + \int_1^3 \left(1 - e^{1-t} \right) dt$

$= \left(-t - e^{1-t} \Big|_0^1 \right) + \left(t + e^{1-t} \Big|_1^3 \right)$

$= (-1 - 1 + e) + \left(3 + e^{-2} - 1 - 1 \right)$

$= e + e^{-2} - 1$

$4 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{antidifferentiation} \\ 1 : \text{evaluation} \end{cases}$

OR

OR

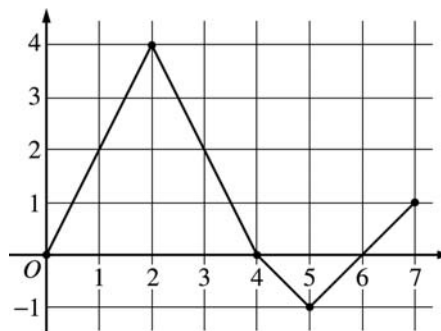$x(t) = -t - e^{1-t}$

$x(0) = -e$

$x(1) = -2$

$x(3) = -e^{-2} - 3$

Distance $= (x(1) - x(0)) + (x(1) - x(3))$

$= (-2 + e) + \left(1 + e^{-2} \right)$

$= e + e^{-2} - 1$

$4 : \begin{cases} 1 : \text{any antiderivative} \\ 1 : \text{evaluates } x(t) \text{ when} \\ \quad t = 0, 1, 3 \\ 1 : \text{evaluates distance} \\ \quad \text{between points} \\ 1 : \text{evaluates total distance} \end{cases}$

Question 5

Let f be a function defined on the closed interval $[0,7]$. The graph of f, consisting of four line segments, is shown above. Let g be the function given by $g(x) = \int_2^x f(t)\,dt$.

(a) Find $g(3)$, $g'(3)$, and $g''(3)$.

(b) Find the average rate of change of g on the interval $0 \le x \le 3$.

(c) For how many values c, where $0 < c < 3$, is $g'(c)$ equal to the average rate found in part (b)? Explain your reasoning.

(d) Find the x-coordinate of each point of inflection of the graph of g on the interval $0 < x < 7$. Justify your answer.

Graph of f

(a) $g(3) = \int_2^3 f(t)\,dt = \frac{1}{2}(4+2) = 3$

$g'(3) = f(3) = 2$

$g''(3) = f'(3) = \dfrac{0-4}{4-2} = -2$

$$3 : \begin{cases} 1 : g(3) \\ 1 : g'(3) \\ 1 : g''(3) \end{cases}$$

(b) $\dfrac{g(3) - g(0)}{3} = \dfrac{1}{3}\int_0^3 f(t)\,dt$

$= \dfrac{1}{3}\left(\dfrac{1}{2}(2)(4) + \dfrac{1}{2}(4+2)\right) = \dfrac{7}{3}$

$$2 : \begin{cases} 1 : g(3) - g(0) = \int_0^3 f(t)\,dt \\ 1 : \text{answer} \end{cases}$$

(c) There are two values of c.

We need $\dfrac{7}{3} = g'(c) = f(c)$

The graph of f intersects the line $y = \dfrac{7}{3}$ at two places between 0 and 3.

$$2 : \begin{cases} 1 : \text{answer of 2} \\ 1 : \text{reason} \end{cases}$$

Note: 1/2 if answer is 1 by MVT

(d) $x = 2$ and $x = 5$

because $g' = f$ changes from increasing to decreasing at $x = 2$, and from decreasing to increasing at $x = 5$.

$$2 : \begin{cases} 1 : x = 2 \text{ and } x = 5 \text{ only} \\ 1 : \text{justification} \\ \quad (\text{ignore discussion at } x = 4) \end{cases}$$

Question 6

Let f be the function satisfying $f'(x) = x\sqrt{f(x)}$ for all real numbers x, where $f(3) = 25$.

(a) Find $f''(3)$.

(b) Write an expression for $y = f(x)$ by solving the differential equation $\dfrac{dy}{dx} = x\sqrt{y}$ with the initial condition $f(3) = 25$.

(a) $f''(x) = \sqrt{f(x)} + x \cdot \dfrac{f'(x)}{2\sqrt{f(x)}} = \sqrt{f(x)} + \dfrac{x^2}{2}$

$f''(3) = \sqrt{25} + \dfrac{9}{2} = \dfrac{19}{2}$

$$3 : \begin{cases} 2 : f''(x) \\ \quad <-2> \text{ product or} \\ \quad \text{chain rule error} \\ 1 : \text{value at } x = 3 \end{cases}$$

(b) $\dfrac{1}{\sqrt{y}}\,dy = x\,dx$

$2\sqrt{y} = \dfrac{1}{2}x^2 + C$

$2\sqrt{25} = \dfrac{1}{2}(3)^2 + C \; ; \; C = \dfrac{11}{2}$

$\sqrt{y} = \dfrac{1}{4}x^2 + \dfrac{11}{4}$

$y = \left(\dfrac{1}{4}x^2 + \dfrac{11}{4}\right)^2 = \dfrac{1}{16}\left(x^2 + 11\right)^2$

$$6 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivative of } dy \text{ term} \\ 1 : \text{antiderivative of } dx \text{ term} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition } f(3) = 25 \\ 1 : \text{solves for } y \end{cases}$$

Note: max 3/6 [1-1-1-0-0-0] if no
      constant of integration

Note: 0/6 if no separation of variables