CollegeBoard
Advanced Placement Program

Student Performance Q&A:

2005 AP® Calculus AB and Calculus BC Free-Response Questions

The following comments on the 2005 free-response questions for AP® Calculus AB and Calculus BC were written by the Chief Reader, Caren Diefenderfer of Hollins University in Roanoke, Virginia. They give an overview of each free-response question and of how students performed on the question, including typical student errors. General comments regarding the skills and content that students frequently have the most problems with are included. Some suggestions for improving student performance in these areas are also provided. Teachers are encouraged to attend a College Board workshop, to learn strategies for improving student performance in specific areas.

AB Question 1/BC Question 1

What was the intent of this question?

This problem gave two functions whose graphs intersected at $x \approx 0.178218$ and at $x = 1.$ Students had to find those first two points of intersection to the right of the *y*-axis. In part (a) students had to find the area of the region bounded by the *y*-axis and the two graphs. In part (b) they had to find the area bounded by the two graphs between the first two points of intersection. Part (c) tested whether students could set up and then evaluate the definite integral that represented the volume of the solid of revolution obtained when the region, whose area was found in part (b), was revolved about the horizontal line $y = -1.$

How well did students perform on this question?

In general, students performed quite well. There were very few blank papers. Most students understood how to set up and determine the two areas in parts (a) and (b). The greatest difficulty in part (c) was that students had practiced finding the volume of solids of revolution when rotating about an axis, but this problem asked them to rotate a region about a horizontal line $y = -1.$ Some students were unprepared to complete this more complicated task.

The mean score was 5.73 for AB students and 7.09 for BC students (out of a possible 9 points). Almost 22 percent of AB students and over 36 percent of BC students received a 9. Approximately 12 percent of AB students and 2 percent of BC students did not earn any points.

What were common student errors or omissions?

Some students used the trace key on the calculator instead of the solve command to determine the intersection points. This did not give intersection points with the desired three-decimal accuracy. Many students forgot to indicate limits on their integrals so did not earn points. However, the most frequent error was difficulty in setting up an integral for the volume of a solid of revolution that involved a rotation about the horizontal line $y = -1.$ Students needed to demonstrate the ability to move between the numerical realm of the calculator and the geometrical setting of the given graph in order to determine the two x-coordinates of the points where the graphs of f and g intersect. Some students were confused because f and g intersect in more than two places, and a few students did not understand that the given region is determined by the two intersection points in the first quadrant that lie closest to the y-axis.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

- Teach students how to determine whether the values they compute on their calculators are reasonable.
- Direct students to express themselves using proper mathematical notation—students must be reminded that what they write is what will be scored, and it may not be what they had *intended* to write.
- Remind students that when they are working on a problem in the part of the exam that requires a graphing calculator, they should actually *use* their calculators rather than attempting to do all of the differentiation or antidifferentiation by hand.
- Continue to require students to practice finding the volume of solids formed by rotating a region about a line other than a coordinate axis.

AB Question 2

What was the intent of this question?

This problem described a situation in which sand was simultaneously being removed from a beach by the tide and replaced by a pumping station. Functions that modeled the rates of removal and replacement were provided, and students were given the initial amount of sand. Parts (a) and (b) tested knowledge of the Fundamental Theorem of Calculus by challenging students to find the total amount of sand removed by the tide over the first six hours and to find an expression, necessarily an integral expression or one obtained through integration, for the total amount of sand on the beach at time $t.$ Part (c) tested whether students could combine the rates correctly to find the rate at which the total amount of sand on the beach was changing. Part (d) tested whether they could then use this combined rate to find the time at which the amount of sand was at its minimum, justify this answer, and use the Fundamental Theorem of Calculus or the solution to part (b) to give the amount of sand at that time.

How well did students perform on this question?

Student performance was a bit weak. In parts (a) and (c) students did well. In part (b) they struggled, because the desired $Y(t)$ involved an integral with a constant lower limit of integration and an independent variable as the upper limit. In part (d) most students earned the first 2 points but had a great deal of difficulty in making a global argument for the minimum value of the function over the given interval.

The mean score was 3.22 (out of a possible 9 points). Less than 10 percent of the students received an 8 or 9, and almost 22 percent of students did not earn any points.

What were common student errors or omissions?

Students had trouble recognizing and then using the given rate functions. They also made errors when they attempted to create an integral expression for the total number of cubic yards of sand on the beach. Many students were unable to choose an appropriate test to determine global extrema.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

- Emphasize that careful and explicit explanations are required to link the sign of the derivative to the behavior of the function. Indicating precise intervals for which properties hold must be done explicitly in interval or inequality notation and cannot be interpreted from a sign chart. Sign charts without accompanying explanatory text cannot stand alone as justification.
- Remind students that when they are working on a problem in the part of the exam that requires a graphing calculator, they should actually *use* their calculators rather than attempting to do all of the differentiation or antidifferentiation by hand.
- Give students practice with similar AP Calculus free-response questions that involve rates and require construction of functions that describe how amounts change over time (see the 2005 Form B question AB2/BC2 and the 2002 question AB2/BC2, for example).
- Give students more practice in finding global extrema and encourage them to test all candidates by comparing the function values at each of the critical points and endpoints to determine the absolute maximum and absolute minimum function values. This method is often much neater and easier than using a derivative test.

AB Question 3/BC Question 3

What was the intent of this question?

This problem presented students with a table of the temperatures, $T(x)$, of a wire taken at intervals of unequal lengths, where x was the distance from the heated end of the wire. Part (a) asked for an estimation of the derivative of T at a point midway between two of the table values. It was expected that students would use the average rate of change of T over this interval to approximate the derivative. Part (b) required setting up an integral for the average temperature and using all of the information in the table to find a trapezoidal approximation of the integral over the length of the wire. Stating correct units for average temperature was an important part of this problem. Part (c) tested student knowledge of the Fundamental Theorem of Calculus. Students had to recognize $\int_0^8 T'(x)\,dx$ as the total change in the value of T between the endpoints at $x = 0$ and at $x = 8$. Part (d) asked whether the tabular data were consistent with a strictly positive second derivative at each point between $x = 0$ and $x = 8$. One way to do this was to find average rates of change over successive intervals of unequal lengths and to recognize that since these average rates of change were not increasing, the second derivative could not be positive on the entire interval. Ideally, students should have invoked the Mean Value Theorem to make the connection between the average rate of change and the value of the first derivative at some point in the interval.

How well did students perform on this question?

The overall performance was poor, with many scores in the 0–2 range and few scores above 6. The large number of 0 scores indicates that many students were not able to get into the problem. Those scoring in the 4–6 range tended to get most of their points in the first three parts but did not earn points in part (d). It is also possible that making this question the last calculator problem contributed to low scores for students who ran out of time on the calculator section and never had a chance to get back to the problem.

The mean score was 1.76 for AB students and 3.32 for BC students (out of a possible 9 points). Approximately 40 percent of AB students did not earn any points, and less than 1 percent of AB students received a 9. Almost 17 percent of BC students did not earn any points, and less than 4 percent of BC students received a 9.

What were common student errors or omissions?

Despite the recent emphasis on multiple ways of presenting a function, many students had difficulty with a tabular form where they were unable to determine an explicit expression. Many students calculated a correct estimate for $T'(7)$ and then used their value inappropriately, so they did not earn the point in part (a). Some students used the formula for a trapezoidal estimate with equal subintervals. Since the subintervals given in this problem were unequal, that formula did not apply. Many students also did not earn the units point because they forgot to include units on their answers in parts (a), (b), and (c). A few students created a continuous function on $[0, 8]$ that matched the data in the table and did not earn points, because they based their answers on properties of the continuous function.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

- Give students more practice with functions that are given numerically at a finite number of domain values. Students must learn that under those circumstances, it is not valid to create and then reason from a continuous function.
- Help students develop a deeper understanding of estimation methods. Place less emphasis on the Trapezoidal Rule (the formula) and more emphasis on the trapezoidal method.

AB Question 4

What was the intent of this question?

In this problem, students were given the values of a continuous function f and its first and second derivatives at four points, together with the signs of these functions in the unit interval to the right of each of the four points. Part (a) tested whether students could use the information in the table to identify the locations of the relative extrema of f and justify their answers. Part (b) asked for a sketch of the function.

Students needed to make the critical observation that if a continuous function is increasing and its graph is concave up to the left of $x = 2$ and decreasing and its graph is concave down to the right of $x = 2,$ then there must be a point of nondifferentiability at $x = 2.$ Parts (c) and (d) tested knowledge of the Fundamental Theorem of Calculus, defining g as the definite integral of f from $t = 1$ to $t = x.$ Students were asked to find and justify the relative extrema of g and the points of inflection of the graph of $g.$

How well did students perform on this question?

Many students were able to use the given information properly in part (a), but others had difficulty. This was the first year of the new sign chart policy, and many students presented arguments that might have received credit in a previous year but were not sufficient according to the new policy. The graph in part (b) was graded gently. Many students had trouble showing correct behavior at $(2, 2).$ In part (c) most students knew to use information about f to determine where g had its extrema. In part (d) many students talked about the change in concavity without explaining the reasons for their statements.

The mean score was 3.48 (out of a possible 9 points). Approximately 5 percent of students received a 9, and 17 percent did not earn any points on this question.

What were common student errors or omissions?

Students had trouble communicating information given to them in a sign chart. Many did not realize that they needed to state how and where the derivative changed sign instead of talking about a change from increasing to decreasing (or concave up to concave down). In part (b) some graphs were discontinuous at $x = 2$ because students did not know how to treat the fact that the derivative did not exist at $x = 2.$ Students could successfully use the Second Derivative Test to show that the function has a maximum at $x = 3.$ Most of these students also tried to use the Second Derivative Test at $x = 1,$ but the Second Derivative Test does not apply there. Students who did not understand how to handle a function defined by a definite integral with a variable limit had difficulty completing parts (c) and (d).

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

- Explain the change in the sign chart policy. Although this was communicated to teachers in various ways (on AP Central, at the 2004 AP Calculus Reading, in College Board workshops, and via the AP participation mailing to schools), it is clear that some students still do not understand the new policy.
- Provide students with opportunities to practice written communication skills, since the AP Calculus Exams are requiring more justifications.
- Give students more practice in working with functions defined by a definite integral with a variable limit. (This question is a good one for teachers to use to help their students learn these skills.)

AB Question 5/BC Question 5

What was the intent of this question?

Students were given a piecewise-linear graph that modeled a car's velocity at time $t.$ Testing the ability to connect integrals with area under a graph, part (a) asked students to find the integral of v from $t = 0$ to

$t = 24$ and to explain the meaning of this integral, namely, that it represented the change in position over that time interval. Part (b) asked students to determine values of the derivative of v from the graph and to indicate units of measure. Part (c) asked for a piecewise-defined function that described the car's acceleration. For part (d) students had to calculate the average rate of change of v and whether the Mean Value Theorem would guarantee the existence of a point on the interval where the derivative took on that value. In parts (b) and (c) students had been led to discover points at which the derivative of v was not defined. This question tested whether they recognized that the Mean Value Theorem does not apply when the derivative of v is not defined over the entire interval.

How well did students perform on this question?

Overall student performance was disappointing. Students have difficulty writing mathematics, and their weaknesses in that area were evident in this problem. Although they were able to use informal terms that indicated some knowledge, very few students were able to express their answers using precise mathematics.

The mean score was 2.90 for AB students and 4.56 for BC students (out of a possible 9 points). Less than 2 percent of AB students received a 9, and approximately 21 percent of AB students did not earn any points. In addition, less than 5 percent of BC students received a 9, and approximately 6 percent of BC students did not earn any points.

What were common student errors or omissions?

Students had a great deal of trouble with the language and concepts involved in explaining that $v'(4)$ does not exist. In addition, they were confused about how to handle $v'(20)$. Students did not show a complete understanding of the Mean Value Theorem as they worked this problem.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

- Provide students with increased opportunities to reason from information presented in graphical form. Encourage them to calculate definite integrals by geometric rather than by analytic means when possible.
- Give students many problems that will help them learn to communicate their ideas using precise mathematical language. The language of calculus includes precise terms for nondifferentiability. Teachers must use the correct terms to describe these situations and should not invent their own language.

AB Question 6

What was the intent of this question?

Students were presented with a separable differential equation and asked to sketch its slope field in part (a). In part (b) students had to know how to use the differential equation to find the equation of the line tangent to the solution curve through the point $(1, -1)$. They were also asked to use this tangent to approximate the value of the particular solution at $x = 1.1$. Part (c) required solving the separable differential equation to find an exact formula for the solution with $f(1) = -1$. There were two branches to the solution, and students had to choose the correct branch.

How well did students perform on this question?

Overall performance was good. Students were able to effectively enter into the problem in one or more of the three parts, regardless of difficulties they may have encountered in other parts. Scores were strong compared to similar problems in previous years. The problem exhibits a nice connection between the three parts, giving students the opportunity to look for consistency in their responses. Most students did not show evidence that they saw these underlying relationships, but this did not hinder them in providing correct work in individual parts of the problem.

The mean score was 3.95 (out of a possible 9 points). Approximately 1 percent of students received a 9, and about 15 percent of students did not earn any points.

What were common student errors or omissions?

In part (a) the slope fields were quite good away from the *x*-axis. Many students had trouble deciding how to handle $\frac{dy}{dx} = -\frac{2x}{y}$ at the origin or struggled with what to do with points on the *x*-axis, even though they were not asked to use these points in their slope field. A few students had trouble using the differential equation to find the equation of the line tangent to the graph of f at $(1, -1)$. Many chose the incorrect branch of $y = \pm\sqrt{3 - 2x^2}$ in their solution to part (c). In addition, algebra and arithmetic errors caused students to lose points.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

- Teach slope fields. Most students were well prepared for this question, although some indicated that they had not studied slope fields. Teachers need to be familiar with the updated topic outlines in the most recent *AP Calculus AB and Calculus BC Course Description*. (This was the second year that slope fields appeared on the Calculus AB Exam.)
- Give students more practice with separable differential equations. Some students could not do the algebra in separating variables. Many came up with an exponential function as their answer in part (c) and were unable to solve for the correct value of the constant of integration. The most frequent errors were due to the lack of algebraic skills.
- Prepare students for the format of the exam and remind them to write their work in the appropriate place for each part of the problem. Many students had the correct work for part (c) in the space for part (b) in the exam booklet.

BC Question 2

What was the intent of this question?

Students were given a curve described in polar coordinates with r as a function of $\theta.$ To avoid an unfair advantage for students with calculators that have a computer algebra system, the derivative of r with respect to θ was also given. Part (a) tested students' ability to find the area of a region bounded by a curve described in polar coordinates. Part (b) tested their ability to convert between rectangular and polar coordinates, giving the *x*-coordinate of a point on the curve and asking for its angle $\theta.$ In part (c) students were asked to explain what the sign of $\frac{dr}{d\theta}$ says about the behavior of the curve. They were expected to recognize that when this derivative is negative, the curve is getting closer to the origin. Part (d) asked students to find the $\theta-$coordinate of the point on the curve farthest from the origin, testing whether they could recognize that this must happen at a point where $\frac{dr}{d\theta}$ changes from positive to negative.

How well did students perform on this question?

Generally, responses indicated that students have at least some facility with working in polar coordinates. A vast majority identified r as the radius or distance from the pole in part (c), and many demonstrated knowledge of the relationship between rectangular and polar coordinates in part (b). However, a smaller percentage of students knew how to compute the area required in part (a).

The mean score was 3.50 (out of a possible 9 points). Approximately 2 percent of students received a 9, and 16 percent of students did not earn any points.

What were common student errors or omissions?

The most difficult part of this question for students was the justification of an absolute maximum value in part (d). Some students did not use the correct setup to find the area in part (a). Instead of using $\frac{1}{2}\int_0^{\pi} r^2\ d\theta$ to find the desired area in part (a), some students used an integral of the form $\int_0^{\pi} r\ d\theta.$ Many students found the value of $\theta = \frac{\pi}{3}$ in part (d) but could not provide the necessary global argument to justify an absolute maximum value.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

- Emphasize the method for finding area in polar coordinates. Many students were unfamiliar with this standard procedure.
- Help students understand the difference between a relative and an absolute argument and the various means by which a global argument can be made. As has been the case on past AP Calculus Exams, students often had difficulty distinguishing between a local argument and a global one when dealing with extrema. Teachers and students should be familiar with the method of testing all candidates, as that is often the easiest way to make an argument about global extrema.

- Explain the relatively new AP Calculus sign chart policy (see the article "On the Role of Sign Charts in AP Calculus Exams for Justifying Local or Absolute Extrema" on AP Central).

BC Question 4

What was the intent of this question?

Students were given a differential equation and asked to sketch its slope field in part (a). In part (b) they needed to show how to use the differential equation to find the minimum value of the solution that has its minimum at $x = \ln\left(\frac{3}{2}\right)$. Part (c) asked students to demonstrate their knowledge of Euler's method by approximating the value of f at $x = -0.4$, where f was the solution that satisfied $f(0) = 1$. Part (d) asked them to find the second derivative of y with respect to x and to use it to determine, with explanation, whether the estimate obtained from Euler's method was necessarily too low or too high.

How well did students perform on this question?

The majority of students were able to sketch the slope field in part (a). In part (b) the concept that $\frac{dy}{dx}$ must equal 0 was an easy one to use once students realized that a function differentiable everywhere must have $\frac{dy}{dx} = 0$ at a local minimum point. Students had to demonstrate that they were doing two steps of Euler's method with $\Delta x = -0.2$ to earn the first point in part (c). Some students clearly did not know Euler's method, but more did not earn the second point due to bad arithmetic. In part (d) most students were able to calculate $\frac{d^2y}{dx^2}$, but few were able to give acceptable reasoning for the conclusion that the approximation from part (c) was less than the actual value of f at $x = -0.4$. Many students gave no reason, but many gave a local or pointwise argument rather than a global one.

This question was split scored with parts (a) and (b) being the AB-only material (sketching the slope field and determining a local minimum value) and parts (c) and (d) the BC-only material (Euler's method and interpreting the result of Euler's method). The mean score on parts (a) and (b) was 3.22 (out of a possible 5 points), with about 25 percent of students receiving a 5 and less than 6 percent earning no points. The mean score on parts (c) and (d) was 1.47 (out of a possible 4 points), with only 0.5 percent of students receiving a 4 and about 25 percent earning no points. The total mean score was 4.68 (out of a possible 9 points).

What were common student errors or omissions?

There were three common errors in part (a). First, some students accidentally left out one or two of the 12 points at which they were to draw a slope field segment. Second, vertical segments were drawn at $(0, 0)$ and $(1, 2)$ rather than horizontal ones. Third, segments were drawn with positive slopes instead of negative slopes, or with negative slopes instead of positive slopes. The most frequent reason for not earning the point for drawing the solution curve was drawing no curve. In part (b) about 5 to 10 percent of students tried to solve the differential equation. This made the problem more difficult for them. A few actually earned both points with their correct solution. However, this really did not help them with part (d). In part (c) many students did not use $\Delta x = -0.2$ in their Euler approximation. Most students lost points in part (d) because they gave a local concavity argument.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

- Remind students to use care when sketching slope fields.
- Give students practice with performing Euler's method calculations without a calculator, since they will be expected to do this on the exam. Assign problems that require students to work with decimals and fractions by hand throughout the year.
- Remind students that according to the general exam instructions, answers (numeric or algebraic) do not need to be simplified. Premature rounding and misusing parentheses were also issues for a good number of students.
- Teach students how to use concavity to determine the nature of an Euler's method estimate.

BC Question 6

What was the intent of this question?

This problem gave the value of f and of all of its derivatives at $x = 2.$ Part (a) asked for the sixth-degree Taylor polynomial about $x = 2.$ Part (b) asked for the coefficient of the general term in the Taylor series for $f.$ Part (c) asked students to find the interval of convergence of this series with explanation. This required finding the radius of convergence, testing for convergence at each endpoint, and centering the resulting interval at $x = 2.$

How well did students perform on this question?

A great many students were not prepared for this problem. Those who were able to enter the problem often earned the three points in part (a) and nothing else. Many students either generalized their answer from part (a), even if it was not correct, or started over and correctly answered part (b). Part (c) was the most difficult part of the problem. While most students who attempted this part used the Ratio Test (and occasionally the Root Test), they subsequently made algebraic mistakes or did not use the correct general term. Many students who did find the interior of the interval of convergence did not know to continue and check the endpoints.

The mean score on this problem was 2.47 (out of a possible 9 points), a bit worse than the score of 2.77 on the 2004 BC series problem. Only 1.4 percent of students received a 9, and approximately 26 percent of students did not earn any points.

What were common student errors or omissions?

Most students did not realize that there was a connection between parts (b) and (c). Many presented three different series in each part of the problem and saw no connections among the three parts. A common error was for answers to consist of a power series in all powers, instead of just even powers. Another common error was that students were unable to justify the convergence or divergence of their series.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

- Emphasize techniques and methods that will give students experience with writing and manipulating power series, particularly ones that have only even or only odd powers. The communication of precise mathematics with standard language and notation is very important in a series problem. Students need to be comfortable using both limit and summation notation accurately.
- Stress algebraic skills, which play an important part in students’ success with series questions.
- Remind students about the importance of correctly presenting and organizing their work—many did not do so on this question. Important ideas often appeared in the corner of the page, and solutions did not follow a logical progression.
- Help students understand all aspects of series problems. Many students wrote only the terms of the series when they were referring to the entire series. Series questions require that students know and understand a variety of convergence tests, and complete arguments must include a reference to appropriate convergence and divergence tests. See the current *AP Calculus AB and Calculus BC Course Description* for a complete list of series topics that BC students should know. In addition, students must present a complete endpoint analysis when asked for the interval of convergence of a given power series.