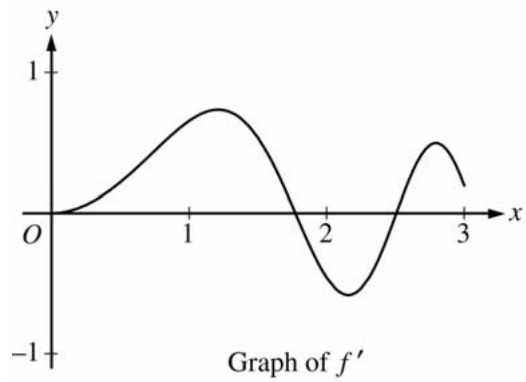
Question 2



Let f be the function defined for $x \geq 0$ with $f(0) = 5$ and f', the first derivative of f, given by $f'(x) = e^{(-x/4)} \sin\left(x^2\right)$. The graph of $y = f'(x)$ is shown above.

(a) Use the graph of f' to determine whether the graph of f is concave up, concave down, or neither on the interval $1.7 < x < 1.9$. Explain your reasoning.

(b) On the interval $0 \leq x \leq 3$, find the value of x at which f has an absolute maximum. Justify your answer.

(c) Write an equation for the line tangent to the graph of f at $x = 2$.

(a) On the interval $1.7 < x < 1.9$, f' is decreasing and thus f is concave down on this interval.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

(b) $f'(x) = 0$ when $x = 0, \sqrt{\pi}, \sqrt{2\pi}, \sqrt{3\pi}, \ldots$
On $[0, 3]$ f' changes from positive to negative only at $\sqrt{\pi}$. The absolute maximum must occur at $x = \sqrt{\pi}$ or at an endpoint.

$f(0) = 5$

$f\left(\sqrt{\pi}\right) = f(0) + \int_0^{\sqrt{\pi}} f'(x)\,dx = 5.67911$

$f(3) = f(0) + \int_0^3 f'(x)\,dx = 5.57893$

This shows that f has an absolute maximum at $x = \sqrt{\pi}$.

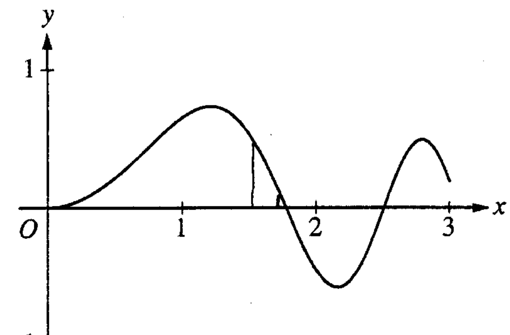
$3 : \begin{cases} 1 : \text{identifies } \sqrt{\pi} \text{ and 3 as candidates} \\ \qquad\text{- or -} \\ \quad\text{indicates that the graph of } f \\ \quad\text{increases, decreases, then increases} \\ 1 : \text{justifies } f\left(\sqrt{\pi}\right) > f(3) \\ 1 : \text{answer} \end{cases}$

(c) $f(2) = f(0) + \int_0^2 f'(x)\,dx = 5.62342$

$f'(2) = e^{-0.5} \sin(4) = -0.45902$

$y - 5.623 = (-0.459)(x - 2)$

$4 : \begin{cases} 2 : f(2) \text{ expression} \\ \quad 1 : \text{integral} \\ \quad 1 : \text{including } f(0) \text{ term} \\ 1 : f'(2) \\ 1 : \text{equation} \end{cases}$



Graph of f'

Work for problem 2(a)

if $f''(x) < 0$ on some interval, then the graph of f is concave down on this interval

if $f''(x) > 0$ on some interval, then the graph is concave ~~down~~ up on this interval

the interval where $f''(x) < 0$ determines also where $f'(x)$ is decreasing, and where $f''(x) > 0$, $f'(x)$ is increasing.

From the graph of $f'(x)$, it can be seen that on the interval $(1.7;1.9)$, $f'(x)$ is decreasing, therefore, the graph of $f(x)$ will be concave down.

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Work for problem 2(b)

First, find the critical points. Critical points are where $f'(x)=0$ or $f'(x)\nexists$

~~From the graph it can be seen that $f'(x)=0$ at $x=0$, x~~

$f'(x)=0$: $e^{(-x/4)}\sin(x^2)=0$: $x=0$, $x=1.77245$, $x=$

$f'(x)$		+	−	+
x	0	1.77245	2.5066	3
$f(x)$	↗	max ↘	min ↗	

Since $f'(x)$ changes from positive to negative at $x=1.77245$, $f(1.77245)$ is a relative maximum. As $f'(x)$ changes from negative to positive at $x=2.5066$, $f(2.5066)$ is a relative minimum.

Since $f'(x)$ is positive on $(0; 1.77245)$, $f(x)$ increases on this interval and $f(1.77245) > f(0)$

So, the absolute maximum is either at $x=1.77245$ or at the upper endpoint $x=3$

$f(1.77245) = f(0) + \int_0^{1.77245} e^{(-x/4)}\sin(x^2)dx = 5+0.6791141 = 5.6791141$

$f(3) = f(0) + \int_0^3 e^{(-x/4)}\cdot\sin(x^2)dx = 5+0.5789285 = 5.5789285$

$f(1.77245) > f(3)$, Thus, $x=1.77245$ is the value of x where f has an absolute maximum

Work for problem 2(c)

$x_0=2$

$y_t = f(x_0) + f'(x_0)(x-x_0)$

$f(2) = f(0) + \int_0^2 e^{(-x/4)}\cdot\sin(x^2)dx = 5+0.6234267 = 5.6234267$

$f'(2) = -0.4590239$

$y_t = 5.6234267 - 0.4590239(x-2)$

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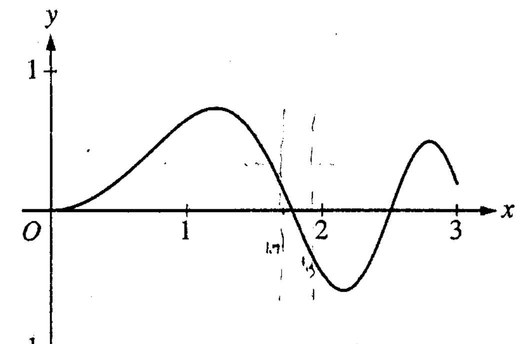

Graph of f'

Work for problem 2(a)

Step 1: Find the inflection point between 1.5 and 2 approx

$$f'(x) = e^{(-x/4)} \sin(x^2)$$

$$f''(x) = -\frac{e^{(-x/4)}}{4} \sin(x^2) + e^{(-x/4)} (2x)\cos(x^2)$$

inflection $\Rightarrow$ $f''(x) = 0$

No inflection pts found between 1.7 & 1.9

$\Rightarrow$ either concave up or down

$\Rightarrow$ $f''(1.7) = -2.1935 < 0$ concave down

to check

$f''(1.9) = -2.0384 < 0$ concave down

Over the interval $1.7 < x < 1.9$ the graph of f is CONCAVE DOWN

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 2B

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Work for problem 2(b)

For f to have an absolute maximimum in the interval $0 \le x \le 3$ there has to exist a point at which f' is ZERO & f'' is < 0 such that it is concave down

From the graph $\Rightarrow$ f' is zero at 3 pts $x = 0$, $x = 1.7724538509$, $x = 2.5066282746$

$\Rightarrow f''(0) = 0$ inflection

$f''(1.7724538509) = -2.276 < 0 \Leftarrow$ Max

$f''(2.5066282746) = 2.679 > 0 \Rightarrow$ Min

x at absolute max. is $x = 1.7724538509$

Work for problem 2(c)

$f'(2) = -0.459$

$x = 2 \qquad y = \int_0^2 e^{(-x/4)} \sin(x^2)\, dx$

$y\big|_{x=2} = -0.6234 + 5 = 4.377$

$y - 4.377 = -0.459(x-2)$

$y = -0.459x + 5.295$

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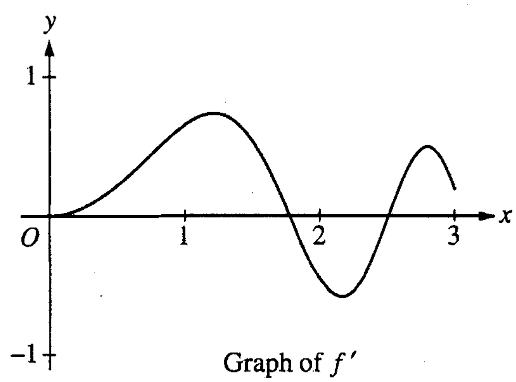
2c



Graph of f'

Work for problem 2(a)

f' is decreasing, therefore $f''(x) < 0$, the function $f(x)$ is concave down

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Work for problem 2(b)

f has a relative max when f' changes its sing from + to -

$X = 1,772$

Then we compare the values of $f(x)$ at $x = 1.772$, $X = 0$, $x = 3$

but as the value of $f'(x)$ on $(0; 1,772) > 0$, that means that $f(1.772) > f(0)$

Work for problem 2(c)

$y = f'(x_0)(x - x_0) + f(x_0)$

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Question 2

Overview

This problem presented students with the graph and the symbolic formula of the derivative of a function f. It explored their understanding of the relationship between the behavior of the derivative and the function. In part (a) students had to know how to determine the concavity of the graph of f by observing where the graph of f' was increasing or decreasing. Part (b) asked students to find and justify the location of the absolute maximum of f over the given closed interval. It was expected that students would use the graphing calculator to find the appropriate critical point for a local maximum and to evaluate f at that critical point and the endpoints using definite integrals. However, students could also find the relevant critical point by hand and compare the values of f at that critical point and at the endpoints using signed areas. Part (c) required knowing the relationship between the values of f' and the slope of the line tangent to the graph of f. It also required the use of a definite integral to compute $f(2)$.

Sample: 2A
Score: 9

The student earned all 9 points. The discussion and conclusion in part (a) is correct. In part (b) the student identifies the appropriate candidates, uses a sign chart to summarize the behavior of the function and the derivative on the interval $[0, 3]$, and then writes a correct explanation for the location of the absolute maximum. The work in part (c) is correct.

Sample: 2B
Score: 6

The student earned 6 points: 2 points in part (a), 1 point in part (b), and 3 points in part (c). In part (a) the student computes the correct second derivative and determines that there are no inflection points between $x = 1.7$ and $x = 1.9$. The student then checks the sign of the second derivative at the two endpoints of the given interval. This is sufficient to make the correct conclusion since there is no inflection point in the interval. In part (b) the student only justifies a local maximum at $x = 1.772$ and thus only earned the third point. In part (c) the student sets up a definite integral for the change in f over the interval from $x = 0$ to $x = 2$ and then adds $f(0)$ in an attempt to compute $f(2)$. This earned the first 2 points. The student also computes the correct derivative value, which earned the third point. The student did not earn the last point because of the error in the evaluation of the integral.

Sample: 2C
Score: 4

The student earned 4 points: 2 points in part (a) and 2 points in part (b). The work in part (a) earned both points. In part (b) the student justifies $x = 1.772$ as the location of a relative maximum. The student indicates the need to compare the value of $f(x)$ at $x = 1.772$ with the values at both endpoints but then only deals with the endpoint at $x = 0$ and not the endpoint at $x = 3$. The student thus earned only the first and third points in part (b). The expression for the tangent line in part (c) is considered a formula and earned no points.