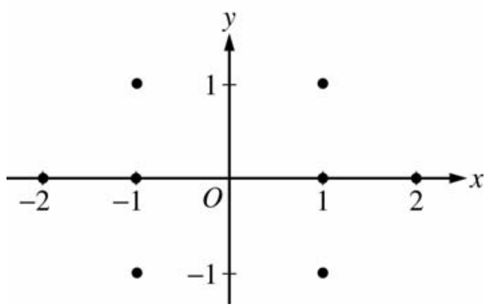
Question 5

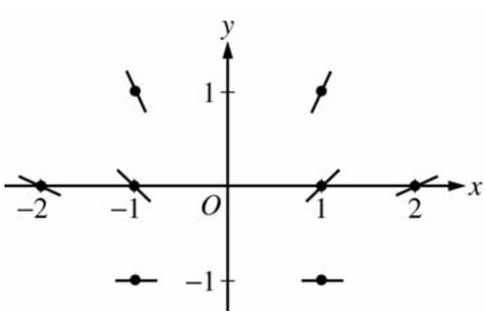
Consider the differential equation $\frac{dy}{dx} = \frac{1 + y}{x}$, where $x \neq 0$.

(a) On the axes provided, sketch a slope field for the given differential equation at the eight points indicated. **(Note: Use the axes provided in the pink exam booklet.)**



(b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(-1) = 1$ and state its domain.

(a)



2 : sign of slope at each point and relative steepness of slope lines in rows and columns

(b) $\frac{1}{1 + y}\, dy = \frac{1}{x}\, dx$

$\ln|1 + y| = \ln|x| + K$

$|1 + y| = e^{\ln|x| + K}$

$1 + y = C|x|$

$2 = C$

$1 + y = 2|x|$

$y = 2|x| - 1$ and $x < 0$

or

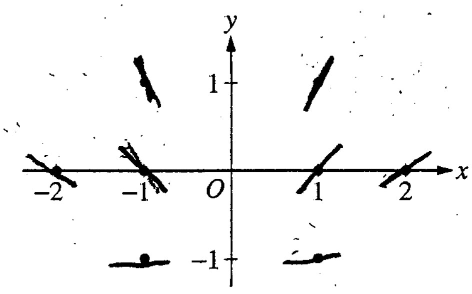
$y = -2x - 1$ and $x < 0$

7 :
- 6 :
  - 1 : separates variables
  - 2 : antiderivatives
  - 1 : constant of integration
  - 1 : uses initial condition
  - 1 : solves for y
- Note: max $3/6$ [1-2-0-0-0] if no constant of integration
- Note: $0/6$ if no separation of variables
- 1 : domain

5A1

NO CALCULATOR ALLOWED

Work for problem 5(a)



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Work for problem 5(b)

$$\frac{dy}{dx} = \frac{1+y}{x}$$

$$\int \frac{dy}{1+y} = \int \frac{dx}{x}$$

$$e^{\ln|1+y|} = e^{(\ln|x|+C)}$$

$1+y = Cx \qquad 1+y > 0$

$y = Cx - 1, \qquad y > -1$

$f(x) = Cx - 1, \qquad f(x) > -1$

$f(-1) = C(-1) - 1 = 1$

$C(-1) = 2$

$C = -2$

$\boxed{f(x) = -2x - 1}, \quad f(x) > -1$

$D = \{x \in \mathbb{R} \mid x < 0\} \qquad -2x - 1 > -1$

or $\qquad -2x > 0$

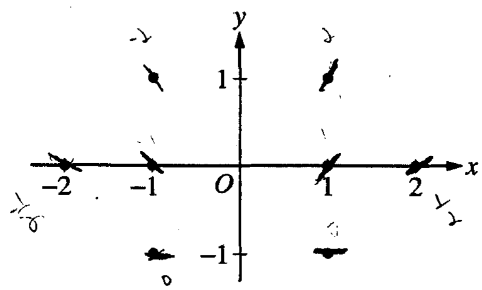
$D = (-\infty, 0) \qquad 2x < 0 \quad x < 0$

GO ON TO THE NEXT PAGE.

5B1

NO CALCULATOR ALLOWED

Work for problem 5(a)



$y' = \frac{1+y}{x}$

$(1,0)$ $\frac{1}{1} = 1$ $m=1$

$(1,1)$ $\frac{2}{1} = 2$ $m=2$

$(1,-1)$ $\frac{0}{1} = 0$ $m=0$

$(2,0)$ $\frac{1}{2} = \frac{1}{2}$ $m=\frac{1}{2}$

$(-1,0)$ $\frac{1}{-1} = -1$ $m=-1$

$(-1,1)$ $\frac{2}{-1} = -2$ $m=-2$

$(-1,-1)$ $\frac{0}{-1} = 0$ $m=0$

$(-2,0)$ $\frac{1}{-2} = -\frac{1}{2}$ $m=-\frac{1}{2}$

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Work for problem 5(b)

$y' = \frac{1+y}{x}$

$\int \frac{dy}{1+y} = \int \frac{dx}{x}$

$\ln_e(1+y) = \ln x + C$ $\qquad 1 = Ce^{\ln(-1)} - 1 \qquad 2 = Ce^{\ln -1}$

$e^{\ln x + C} = 1+y \qquad C = \frac{2}{e^{\ln(-1)}}$

$y = Ce^{\ln x} - 1$ where $x \neq 0$ $\qquad y = \frac{2}{e^{\ln(-1)}}(e^{\ln x}) - 1$

$Ce^{\ln x} \neq 1 \qquad x = 0$

domain: $(-\infty, 0)(0, \infty)$

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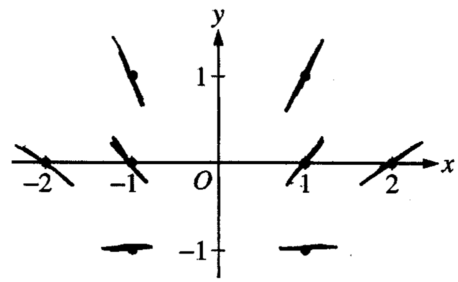
5C1

NO CALCULATOR ALLOWED

Work for problem 5(a)



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5C2

Work for problem 5(b)

$u = 1+y$
$du = 1\,dy$

$\int \frac{1}{1+y}\,dy = \int x\,dx$

$\int \frac{1}{u}\,du = \int x\,dx$

$\ln u = \frac{1}{2}x^2$

$\ln 1+y = \frac{1}{2}x^2$

$\ln 1-1 = 1/2$

$\ln 2 = 1/2$

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2006 SCORING COMMENTARY

Question 5

Overview

This problem presented students with a separable differential equation. In part (a) students were asked to sketch its slope field at eight points. Part (b) required solving the separable differential equation to find the particular solution with $f(-1) = 1$. Students were also asked for the domain of the solution to this differential equation. This is an important consideration when solving any differential equation and in particular when the differential equation is not defined for all values of the independent and/or dependent variables. Students needed to recognize for this equation that the particular solution must be a differentiable function on an open interval that contains $x = -1$ and does not contain $x = 0$.

Sample: 5A
Score: 9

The student earned all 9 points. The statement $y + 1 > 0$ is used by the student to explain why $\ln|1 + y|$ can be replaced by $\ln(1 + y)$.

Sample: 5B
Score: 6

The student earned 6 points: 2 points in part (a) and 4 points in part (b). The student's slope field in part (a) is correct and earned 2 points. The computations were ignored. In part (b) the separation point was earned with the student's second line. The next line earned 1 antidifferentiation point for $\ln(1 + y)$ and no points for $\ln(x)$. The constant of integration point was also earned. The initial condition point was earned in the work on the right side of the page. The $\ln(-1)$ prevented the student from being eligible for the solution point. The domain point was not earned.

Sample: 5C
Score: 3

The student earned 3 points: 2 points in part (a) and 1 point in part (b). The student's slope field in part (a) is correct and earned 2 points. In part (b) the incorrect separation of variables did not earn a point. Since the separation is incorrect, the student was eligible for only 1 antidifferentiation point if both antidifferentiations were correct. Here that occurs. However, the y antidifferentiation is not finished until $\ln(1 + y)$ first appears, and this earned the point. The absence of a constant of integration prevented the student from earning further points. The incorrect separation also implies that the student was not eligible for the solution point. If the constant of integration point had been earned, then the initial condition point would not have been earned since $1 + y \neq -1$ if $y = 1$.