Student Performance Q&A:

2007 AP® Calculus AB and Calculus BC Free-Response Questions

The following comments on the 2007 free-response questions for AP® Calculus AB and Calculus BC were written by the Chief Reader, Caren Diefenderfer of Hollins University in Roanoke, Virginia. They give an overview of each free-response question and of how students performed on the question, including typical student errors. General comments regarding the skills and content that students frequently have the most problems with are included. Some suggestions for improving student performance in these areas are also provided. Teachers are encouraged to attend a College Board workshop to learn strategies for improving student performance in specific areas.

We would like to remind teachers that free-response questions are designed to give students the opportunity to demonstrate their knowledge of calculus topics in a wide variety of contexts. Questions are intentionally written to be both unique and varied from year to year. Reviewing free-response questions from previous AP Calculus Exams is a good practice to familiarize students with the style of questions that are asked, but teachers and students should not assume that the same topics, techniques, and contexts will be tested year after year. This document attempts to help teachers focus on general principles that will improve student performance on future AP Calculus Exams.

Question AB1/BC1

What was the intent of this question?

This problem presented students with a region bounded above by the graph of a function and below by a horizontal line. Because no picture was provided, students were expected to graph the function on their calculators or use their knowledge of rational functions to sketch the graph, and then identify the appropriate region from their graph. The points of intersection of the graph and the horizontal line could be found either algebraically or with the calculator. Students needed to find, in part (a), the area of the region; in part (b), the volume of the solid generated when the region was rotated about the x-axis; and in part (c), the volume of the solid above the region for which the cross sections perpendicular to the x-axis were semicircles.

How well did students perform on this question?

Students who used the correct region did very well on this problem. Most errors resulted from using incorrect integrands in parts (b) and (c); the standards for these integrands were strict. Some students thought that the correct region had a vertical asymptote and was unbounded. Other students thought that

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the region was bounded above by the horizontal line and bounded below by the rational function; this region is, of course, unbounded. Still other students thought that the region was bounded above by the minimum of the two given functions and below by the x-axis, and this region is also unbounded.

The mean score for this question was 4.33 for the AB students and 5.43 for the BC students out of a possible 9 points. About 6.8 percent of the AB students and 13.7 percent of the BC students earned all 9 points. Approximately 15.5 percent of the AB students and 6.8 percent of the BC students did not earn any points.

What were common student errors or omissions?

For a variety of reasons, students worked with regions other than the one we expected. Some of these regions were unbounded. Other students ended up with regions in the first quadrant or with incorrect limits of integration. Even though the stem of the problem described the region in a clear and unambiguous fashion, many students were unable to use their graphing calculators to obtain a correct graph of the intended region.

Some students could not determine the appropriate integrands in parts (b) and (c). Although they recognized that they needed the difference of squares in one part and the square of a difference in the other, they did not use a correct integrand in the appropriate part. Many students also had difficulty determining the correct constant in part (c).

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

- Students must learn to read the free-response questions thoroughly and carefully.
- In a question involving area and volume, students should be careful to find a viewing window on their graphing calculators that gives a complete graph.
- Students should be careful to write what they mean when they give the setup of a definite integral. Some students had a correct final answer but never presented the correct integral, while other students presented the correct integral in written form but were unable to enter it into their calculators correctly.

Question AB2/BC2

What was the intent of this question?

This problem presented students with two functions that modeled the rates, in gallons per hour, at which water entered and left a storage tank. The latter function was piecewise-constant. Graphs of each function were provided. In part (a) students had to use a definite integral to find the total amount of water that entered the tank over a given time interval. Part (b) measured their abilities to compare the two rates to find, with justification, the time intervals during which the amount of water in the tank was decreasing. This could be determined directly from the graphs and the information given about the points of intersection, but students needed to be able to handle the point of discontinuity in the piecewise-defined function. Part (c) asked for the time at which the amount of water was at an absolute maximum and the value of this maximum amount to the nearest gallon. Again, dealing with the critical point at the

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discontinuity was an important part of the analysis, as was using the net rate of change during the first three hours and during the last four hours to compute the total amount of water in the tank at $t = 3$ and $t = 7$, respectively.

How well did students perform on this question?

In part (a) students did fairly well, often earning both the point for the integral and for the answer. In part (b) most students were able to present the correct intervals but had more difficulty earning the reason point. In part (c) students had the most difficulty with justifying the existence of an absolute maximum.

The mean score for this question was 3.03 for the AB students and 4.38 for the BC students out of a possible 9 points. About 2.4 percent of the AB students and 6.6 percent of the BC students earned all 9 points. Approximately 14.7 percent of the AB students and 4.5 percent of the BC students did not earn any points.

What were common student errors or omissions?

In part (a) the most common error involved the misuse of the initial condition. It was not necessary to consider the initial condition in this part of the problem. Some students also used $f(t) - g(t)$ as their integrand instead of recognizing that the correct integrand is $f(t)$.

In part (b) many students had difficulty distinguishing between the *amount* of water that entered or left the tank in a given period of time and the *rate* at which it entered or left. Some students wrote statements that indicated they did not know that f and g are rate-of-change functions.

In part (c) the most common error was to leave out consideration of one or both endpoints in the global analysis. Some students attempted to make arguments without calculating the amount of water at $t = 7$. Some students successfully argued that the absolute maximum was at $t = 3$, but they earned only 4 out of 5 points because they did not calculate the amount of water in the tank at $t = 3$. Some students lost the integrand point because they considered only the rate in $(f(t))$ and not the rate out (250 and 2000, respectively) when computing the net change of water in the tank over the first three hours and over the last four hours.

Even though students were asked to give answers rounded to the nearest gallon in parts (a) and (c), they often presented answers to one or more decimal places. Some students rounded preliminary results in part (c), and as a result their answers were not accurate.

Some students were troubled by the fact that g is not defined at $t = 3$.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

- Teachers need to help students learn to use clear, unambiguous language in their free-response solutions. Students should be able to recognize the difference between amounts and rates.
- Teachers need to work with students on calculator usage, specifically avoiding intermediate rounding and presenting final answers to the required number of decimal places.

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- Students need to be able to make a global extrema argument on a closed interval by the method of testing all candidates, including both critical points and endpoints.

Question AB3

What was the intent of this question?

This problem presented students with a table of selected values of functions f and g, and their first derivatives. A third function h was defined in terms of the composition of f and g. Parts (a) and (b) assessed students' abilities to use the chain rule, the Intermediate Value Theorem, and the Mean Value Theorem to explain why there must be values r and c in the interval $(1, 3)$ where $h(r) = -5$ and $h'(c) = -5$. In part (c) students were given a function w defined in terms of a definite integral of f where the upper limit was $g(x)$. They had to use the Fundamental Theorem of Calculus and the chain rule to calculate the value of $w'(3)$. Part (d) asked them to write an equation for a line tangent to the graph of the inverse function of g at a given value of x. In all parts of this problem students had to use appropriate values from the given table to do their calculations.

How well did students perform on this question?

In general, students performed quite poorly on this question. The most commonly earned points were the first point in part (a) (for computing that $h(1) = 3$ and $h(3) = -7$), the first point in part (b) (for computing the correct difference quotient), and the first point in part (d) (for noting that $g^{-1}(2) = 1$). In part (c) students had a difficult time combining both the chain rule and the Fundamental Theorem of Calculus, and in part (d) they had difficulty finding the derivative of the inverse of a function evaluated at a specific point.

The mean score for this question was a disappointing 0.96 out of a possible 9 points. Only 0.3 percent of the students earned all 9 points. Approximately 59.5 percent of the students did not earn any points.

What were common student errors or omissions?

In part (a) the most common error was the failure to present and/or justify that the hypotheses of the Intermediate Value Theorem were satisfied for the function h. Typically, students noted that the functions f and g were continuous and/or differentiable but failed to tie this to the continuity and/or differentiability of the function h.

A common error in part (b) was the application of the Intermediate Value Theorem to h'. The information in the stem of the problem does not guarantee that the function h' is continuous and thus it is not valid to apply the Intermediate Value Theorem to h'. As in part (a), students often failed to justify that the hypotheses of the Mean Value Theorem were satisfied for the function h. In addition, students occasionally stated an incorrect theorem as justification or called the Mean Value Theorem by some other incorrect name.

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The most common error in part (c) was an incorrect application of the chain rule and/or the Fundamental Theorem of Calculus. The most common incorrect intermediate responses were $w'(x) = f(g(x))$, $w'(x) = f(g(x)) - f(1)$, and $w'(x) = f'(g(x))g'(x)$.

A common error in part (d) was interpreting $g^{-1}(x)$ as the reciprocal of $g(x)$ and proceeding to differentiate this expression by using the power or quotient rule. Many students also used $g(x)$ instead of the inverse, $g^{-1}(x)$, to determine a point on the tangent line. In particular, these students reasoned that since $g(2) = 3$, then $g^{-1}(3) = 2$, and consequently used the point $(3, 2)$. They should have used the fact that $g(1) = 2$ and so $g^{-1}(2) = 1$, making $(2, 1)$ the point to be used.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

- A common error that students made in parts (a) and (b) was attempting to apply a theorem without offering any justification that the hypotheses had been satisfied. Teachers should emphasize to their students that verifying hypotheses is an important step when using a theorem.
- Very few students were successful in part (c) because this problem required the simultaneous use of the Fundamental Theorem of Calculus and the chain rule. Helping students synthesize the most important concepts in the course is essential.
- Many students in part (d) assumed that the notation $g^{-1}(x)$ referred to the reciprocal of $g(x)$, which indicates they need to review the concept of an inverse function and the associated notation.
- In part (d) some students relied on the slope-intercept form of a line, which was a less efficient way to solve this problem. Even though students (and teachers) may have favorite methods, they need to master several approaches to a problem so that they can choose the most efficient technique.
- When a question presents several different functions, students need to clearly articulate the function to which they are referring. In particular, phrases such as “the function” and “the graph” are not precise enough and students may lose points for ambiguous language.

Question AB4

What was the intent of this question?

This problem presented students with a function $x(t)$ describing the position of a particle at time t moving along the x-axis over a closed time interval. Part (a) asked for the time, with justification, when the particle was farthest to the left during this time interval. The first derivative of $x(t)$ was required to compute the time and complete the justification. Part (b) required students to substitute the function and the first and second derivatives of $x(t)$ into the equation $Ax''(t) + x'(t) + x(t) = 0$ to find the value of A. Students did not have to solve the differential equation to determine the value of A.

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How well did students perform on this question?

Students generally started out well with part (a) by differentiating $x(t) = e^{-t} \sin t$ and setting it equal to 0. Many students, however, failed to find the value of t that produces the leftmost position of the particle. An even greater number of students were unable to earn the justification point because they did not use a global argument on the closed interval and instead completed only a local analysis. In part (b) many students were able to make the initial substitution, but errors of differentiation and simplification often prevented them from finding a value for A.

The mean score for this question was 2.91 out of a possible 9 points. About 1.2 percent of the students earned all 9 points. Approximately 33.0 percent of the students did not earn any points.

What were common student errors or omissions?

Many students did not use the product rule, and quite a few students had derivatives that consisted of a single term.

Students made sign errors in differentiating $\sin x$, $\cos x$, and e^{-t}. Some of these were chain rule errors involving the exponent in e^{-t}, and others were the result of not knowing when to include or exclude a negative sign in taking the derivative of a trigonometric function.

Student solutions included incomplete or missing analysis for the absolute minimum. Very few students successfully completed the justification in part (a). They either stated the value of t without any support or appealed to a local argument.

Some students were unable to progress beyond the point of setting $x'(t) = 0$ in part (a). Quite a few students either stopped at this step or selected values for t that were not zeros of the derivative. It was very common for students to select $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$ in addition to, or in place of, the correct critical numbers, $\frac{\pi}{4}$ and $\frac{5\pi}{4}$. Since the derivatives included a few negative signs, it was sometimes difficult to determine where the errors occurred.

Several incorrect answers in part (b) were caused by the misuse or absence of parentheses in the substitution step. Unfortunately, most errors in applying parentheses and collecting like terms resulted in equations that could not be solved.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

- When working with extrema problems, teachers should emphasize the distinction between local and global arguments. It is important to explain the difference between the use of the First Derivative Test and the Second Derivative Test for relative extrema and the procedure to determine an absolute extrema on a closed interval, guaranteed to exist by the Extreme Value Theorem. Students might benefit from seeing more examples where a method of testing all candidates and comparison are required. Teachers should help students practice a coherent and complete process for determining an absolute extrema on a closed interval by evaluating candidate values.

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- Students need to organize their work and use clear notation. Many students did not earn points on this question simply because they did not have a precise and organized procedure. Teachers might help students by insisting that they show all of their work when they are solving algebraic equations.

Question AB5/BC5

What was the intent of this question?

The problem presented students with a table of values for the rate of change of the radius of an expanding spherical balloon over a time interval of 12 minutes. Students were told that the radius was modeled by a twice-differentiable function whose graph was concave down. Part (a) asked students to use a tangent line approximation to estimate the radius of the balloon at a specific time and to determine if the estimate was greater than or less than the true value. This tested their ability to use the information about the concavity of the graph of the radius to make the appropriate conclusion about the behavior of the tangent line. In part (b) students had to handle the related rate of change of the volume, given information about the rate of change of the radius. In part (c) students had to recognize the definite integral as the total change, in feet, of the radius of the balloon from time $t = 0$ minutes to time $t = 12$ minutes and approximate the value of this integral using a right Riemann sum and the data in the table. Part (d) asked students to decide if this approximation was greater than or less than the true value of the definite integral. Again, they were required to use the information about the concavity of the graph of the radius to make the appropriate conclusion about the behavior of the graph of the derivative. Units of measure were important in parts (b) and (c).

How well did students perform on this question?

Students did reasonably well on part (a). Most BC students did well on part (b), but many AB students were unable to take the derivative of $\frac{4}{3}\pi r^3$ with respect to t. Results in part (c) were average, and scores were low in part (d).

The mean score for this question was 2.48 for the AB students and 4.60 for the BC students out of a possible 9 points. About 0.7 percent of the AB students and 3.3 percent of the BC students earned all 9 points. Approximately 35.2 percent of the AB students and 10.7 percent of the BC students did not earn any points.

What were common student errors or omissions?

In parts (a), (c), and (d) students were often quite vague in their discussions. Phrases such as “the graph,” “the function,” and “the slope,” rather than “the graph of r,” “the function $r'(t)$,” and “the slope of $r'(t)$” occurred frequently. Students did not earn points when their language was ambiguous. Many students substituted the value $t = 5$ when they were supposed to use the information that $r(5) = 30$. In part (c) a significant numbers of students set up a Riemann sum with intervals of uniform length, instead of using the data in the given table. Numerous students confused the idea of the definite integral representing the change in the radius over a 12-minute time interval with the value of the radius when $t = 12$.

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Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

- Teachers should help students develop the skill of writing in a precise and accurate manner.
- Teachers should encourage students to write short, precise answers. In part (c) very few students wrote $\int_0^{12} r'(t)\,dt = r(12) - r(0)$, which is the change in the radius, in feet, over the 12-minute time interval.

Question AB6

What was the intent of this question?

This problem presented students with a function that contained a parameter k. In part (a) students had to find the first and second derivatives of the function, making the distinction between the parameter and the variable. Parts (b) and (c) involved finding values of k so that the function or its graph would satisfy certain properties. In part (b) students had to find the value of k for which the function had a critical point at $x = 1$, and then determine whether the function had a relative minimum, relative maximum, or neither at this critical point. In part (c) they were told that the graph of the function had a point of inflection on the x-axis for a certain value of k and were asked to find that value. The x-coordinate of the point of inflection was not given, so students had to write and then solve two nonlinear equations to determine the value of k (and possibly the value of x). Because the problem stated that a point of inflection existed, students were not required to justify that the k value they found actually produced a point of inflection of the graph of the function.

How well did students perform on this question?

In part (a) many students did fairly well. Students who lost only 1 point lost that point in one of two ways: they earned the first point but did not earn the second point because of an algebraic error, or they did not earn the first point because of an error when taking the derivative of $\ln x$. In this second case, students earned the second point by correctly taking the derivative of their first derivative. Those who did not earn any points in part (a) did not take either derivative correctly or were unsuccessful in simplifying correct derivatives. In part (b) most students earned the first 2 points. Many earned the third point as well, but few gave adequate justification to earn the fourth point. Students had trouble earning more than 1 point in part (c). Most students earned the first point but had trouble with the algebra of solving a system of equations. Most students did not make a substitution, so they did not earn the second point and were unable to determine an answer.

The mean score for this question was 3.49 out of a possible 9 points. About 1.4 percent of the students earned all 9 points. Approximately 16.7 percent of the students did not earn any points.

What were common student errors or omissions?

The most common errors in part (a) were computing an incorrect first derivative for $\ln x$ or computing an incorrect derivative for $\frac{1}{x}$. In part (b) the most common error or omission was the justification. Many students merely gave the answer “minimum” without any justification. Students who attempted to justify

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with a First Derivative Test often used only a sign chart or referred to f instead of f' or were unclear in their writing. In part (c) the most common error was using $f''(x) = 0$ only. Many students did not make the substitution that came from setting $f(x) = 0.$ Most students who made this omission chose to substitute $x = 1$ and found an incorrect value for k.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

- Teachers should require students to differentiate as well as antidifferentiate with and without the calculator. Students need more practice finding standard derivatives by hand.
- Teachers should ask for written justification on tests and not accept a sign chart as justification.
- Teachers should encourage students to read the problem carefully and then reread the problem after solving it in order to check that their work is complete.
- Teachers need to emphasize that if two variables are present, students need to determine two equations in order to solve for the two numerical values.

Question BC3

What was the intent of this question?

This problem presented students with two curves described in polar coordinates with r as a function of $\theta.$ The values of θ for which the two curves intersect were also given. Part (a) judged students' ability to find the area of a region bounded by curves described in polar coordinates. Parts (b) and (c) involved the behavior of a particle moving with nonzero velocity along one of the polar curves (and with constant angular velocity $\frac{d\theta}{dt} = 1,$ although students did not need to know that to answer the questions). Students were asked to compute $\frac{dr}{dt}$ and $\frac{dy}{dt}$ at a specific value of $\theta,$ and then to interpret their answers in terms of the motion of this particle. The interesting aspect of the motion was that at this instant, the distance of the particle from the origin was decreasing while its vertical distance from the x-axis was increasing.

How well did students perform on this question?

Many students did not perform well in part (a) because of an inability to apply the proper technique for finding area in polar coordinates, an inability to apply that technique to relevant sections of the given region, or errors made in combining areas of various sections of the desired region. Generally, students were more successful with finding the required derivative in part (b). However, many students did not earn the point for interpreting the derivative at $\theta = \frac{\pi}{3}$ because they did not address the motion of the particle as required. In part (c) students did slightly better when interpreting $\frac{dy}{dt},$ perhaps because it is expressed in the more familiar rectangular coordinates. Students also did fairly well writing y as a function of θ and computing the required derivative, though quite a few students made errors when finding the derivative analytically.

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The mean score for this question was 2.84 out of a possible 9 points. About 2.1 percent of the students earned all 9 points. Approximately 18.2 percent of the students did not earn any points.

What were common student errors or omissions?

In part (a) some students did not know to use $\frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$ to compute area in polar coordinates. For example, some students used $\int_{\alpha}^{\beta} (2 - (3 + 2\cos\theta)) \, d\theta,$ which suggested they were trying to apply techniques that are appropriate for rectangular, rather than polar, coordinates. Students also had difficulty attempting to find the area between two curves, which was needed in some approaches but not necessary when using other approaches to the problem. For example, many students incorrectly considered $\int_{2\pi/3}^{4\pi/3} (2 - (3 + 2\cos\theta))^2 \, d\theta$ rather than $\int_{2\pi/3}^{4\pi/3} \left(2^2 - (3 + 2\cos\theta)^2\right) d\theta$ to find the area between the graphs of $r = 2$ and $r = 3 + 2\cos\theta$ and between the rays $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$. Students also made many conceptual errors in determining which sections enclosed by the curves could be combined to obtain the area of the shaded region.

In the interpretation of the derivatives in parts (b) and (c), the most common error students made was in not addressing the particle's motion as instructed. In part (b) specifically, many stated that the radius is decreasing, which does not address the particle's motion. In part (c) students made many algebraic and arithmetic errors in simplifying and evaluating their expression for $\frac{dy}{dt}$, which is unfortunate since this question was on the calculator-active part of the exam.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

- Note that in this problem the value of r is always positive and the value of y is positive when $\theta = \frac{\pi}{3}$ (part (c)). Teachers should help students see that the interpretations given in the scoring guidelines address both the sign of the derivative and the sign of the variable because these are needed for a correct interpretation. For example, if r were negative for the given value of θ, then a negative value of $\frac{dr}{dt}$ would indicate that the particle is moving away from the origin.

  Similarly, if the value of y were negative for a given θ, a positive value of $\frac{dy}{dt}$ would indicate that the particle is moving closer to the x-axis.
- Teachers should strive to increase the precision with which students describe calculus. The subtlety of this problem was that even though the particle was moving closer to the origin, the y-component of its position was increasing. Students who made statements that contradicted one of these two facts did not earn the interpretation point, even if other parts of their interpretations were correct. Also, teachers should encourage students to describe a value of the derivative as an increase or decrease in the quantity being considered, rather than just a change.

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- Teachers should encourage students to use estimation to check their answers. Many students computed areas in part (a) that were significantly greater than the area of the circle in which the desired region is enclosed. These students might have reconsidered their work if they had realized their answer was not a possible solution to the problem.
- Teachers should strive for their students to build conceptual understanding of concepts rather than a connection of topics with formulas. Student performance on this question might indicate that many students have memorized a formula for area in polar coordinates but did not apply it correctly since they did not use r^2 in the integrand or forgot to multiply by $\frac{1}{2}$.

Question BC4

What was the intent of this question?

This problem presented students with the derivative of a function and an initial value. In part (a) students had to use the given information to write an equation for the tangent line at the initial value. In part (b) they needed to determine whether the graph of the function was concave up or concave down on a given interval and to justify their answer. Part (c) asked students to use antidifferentiation to find an explicit formula for the function. This involved using integration by parts, a BC-only topic.

How well did students perform on this question?

Students did very well in part (a). In part (b) most students used the product rule correctly to find f''. This seemed to cause few problems. However, justifications for the reason were somewhat vague, though many students did have nicely written explanations for the concavity. The work for part (c) was mixed. Many students knew integration by parts well and seemed to go comfortably through the steps. Others made a difficult choice for u and dv, making the subsequent steps more challenging. Still others confused the method in some manner. Almost all attempted the antidifferentiation; there were very few blank papers. Most students attempted each part of the problem.

This question was split scored, with parts (a) and (b) being the AB-material (tangent line and concavity) and part (c) being the BC-only material (integration by parts). The mean score on parts (a) and (b) was 3.98 out of a possible 5 points. About 56.3 percent of the students earned all 5 points, and about 6.6 percent of the students did not earn any points. The mean score on part (c) was 1.80 out of a possible 4 points. About 22.6 percent of the students earned all 4 points, and about 32.9 percent of the students did not earn any points. The mean score for the entire problem was 5.77 out of a possible 9 points. About 17.1 percent of the students received all 9 points, and about 5.5 percent of the students did not earn any points.

What were common student errors or omissions?

Few errors were found in part (a). A relatively small number of students did not evaluate the derivative to find a numerical slope, and a few made errors in simplification.

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In part (b) there were many cases where the work and the explanations were vague or unclear. The prevalence of several notable intervals contributed to this. The interval in the stem of the problem was $x > 0$, while the interval in this part of the problem was $1 \le x \le 3$, and only for $x > \frac{1}{\sqrt{e}}$ is $f''(x) > 0$. This combination led to some incorrect statements when referring to the behavior of f or f''.

Because of the phrasing of the problem, some students assumed that the concavity of f did not change on the interval $(1, 3)$. These students simply evaluated the second derivative at one point, usually $x = 2$ or $x = e$, and used that value to determine the concavity of the graph of f on the entire interval. While a decision was made to accept that explanation because students might have assumed the answer had to be either concave up or concave down, in general this is not an appropriate way to check concavity over an interval (that might contain a point of inflection). We did insist that students make some connection to the interval in question, and most did so, even if the connection was quite loose.

In part (c) some students had difficulty with the mechanics of integration by parts. Many students stopped when they found an antiderivative, forgetting to substitute $f(e) = 2$ to find the specific value of the constant C.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

- For the equation of a tangent line, the point-slope form is perfectly acceptable. Some students relied on the slope-intercept form of a line, which was a less efficient way to solve this problem. Even though students (and teachers) may have favorite methods, they need to master several approaches to a problem so that they can choose the most efficient technique.
- Teachers should remind students that if they use a sign chart, they must explain what is significant in it. In part (b) some students used sign charts, and some students assumed that these would suffice as their justification. More importantly, students need much more practice in writing good mathematical explanations.
- Teachers should review with students the technique of integration by parts as part of preparation for the AP Exam.

Question BC6

What was the intent of this question?

This problem dealt with Taylor series. Part (a) assessed students' abilities to find the first four nonzero terms and the general term of the Taylor series for $f(x) = e^{-x^2}$. Although it would be possible to do this by computing derivatives of the function f, it was expected that students would start with the known Taylor series for the exponential function and use substitution. Part (b) asked for a limit of an indeterminate form $\left(\frac{0}{0}\right)$ involving the function f. Students were asked to use their answer about the Taylor series for f rather than using repeated applications of L'Hospital's Rule. Part (c) required students to formally manipulate the Taylor series for f in a way that could be used to estimate the value of a

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definite integral. Part (d) asked students to explain why the value of the estimate differed from the actual value of the definite integral by less than $\frac{1}{200}$. This question tested whether students could correctly use and justify the error bound for an alternating series whose terms are decreasing in absolute value to zero.

How well did students perform on this question?

In part (a) students who modified the Taylor series for e^x centered at 0 scored well on the first 2 points. Students who tried to generate a Taylor series from the definition of a Taylor series, $\sum_{k=0}^{\infty} \frac{f^{(k)}(c)(x-c)^k}{k!}$, had difficulty due to the complicated higher order derivatives of e^{-x^2}. The general term in part (a) was problematic for many students, and thus many did not attempt to write the general term. Most, if not all, students who based their general term on the derivatives of e^{-x^2} failed to identify the correct general term. Many students who used the Taylor series for e^x centered at 0 to find their general term fared better, but often errors in algebra led to errors in the general term.

In part (b) students who imported an answer of the form $1 - x^2 \pm kx^4 + \cdots$ did well. However, many students failed to distribute the negative sign through the parentheses and reported an answer that was the opposite of the answer for their terms. Students who neglected to follow the directions in part (b) did not receive credit. The intent of the problem was for students to recognize that at an interior point of the interval of convergence, a function can be replaced by its power series representation.

In part (c) students who saw the relationship between parts (a) and (c) by importing their answer from part (a) into part (c) did well. However, many students failed to antidifferentiate their answer from part (a) or made errors when antidifferentiating their polynomial terms from part (a). Some students ignored the statement of the problem and made unsuccessful attempts to evaluate an integral containing e^{-x^2} without using a Taylor series.

Students performed poorly on part (d). Many recognized that the error of the estimate was related to the first omitted term but failed to mention the properties of this alternating series that allowed them to use the first omitted term as the error bound. Some students made futile attempts in part (d) to appeal to the Lagrange error bound.

The mean score for this question was 2.28 out of a possible 9 points. About 0.3 percent of the students earned all 9 points. Approximately 40.6 percent of the students did not earn any points.

What were common student errors or omissions?

In part (a) many students did not use the Taylor series for e^x centered at 0 as a starting point. Some students failed to write a general term or wrote an incorrect general term by not including a negative sign inside parentheses, by squaring the negative sign, by beginning the series with a negative term, or by writing an exponent that resulted in the opposite of the correct series.

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In part (b) many students tried to use L'Hospital's Rule to simplify the limit rather than using their answer from part (a) as instructed in the question. Some students who did use their series from part (a) failed to distribute the negative sign through the parentheses when subtracting the Taylor series for f.

In part (c) several students incorrectly antidifferentiated one or more of the polynomial terms from part (a). In some cases, students failed to include the required four terms.

The second point of (d) was much more problematic for students than the first point. Many students tried to use some form of the Lagrange error bound but were unable to determine this error bound. Most students recognized that the series was an alternating series and the maximum error was bounded by the first omitted term, but they were unable to cite the theorem, or completely cite the hypotheses of the theorem, that guarantees a bound on the estimate. Many students simply stated that they were working with an alternating series and then attempted to explain the size of the error bound; these students did not earn the explanation point.

Based on your experience of student responses at the AP Reading, what message would you like to send to teachers that might help them to improve the performance of their students on the exam?

- Teachers should emphasize that it is important to follow instructions. For example, when asked to write four terms of a Taylor series, students should not elect to simply write the sigma notation without listing the four terms. When asked to use an answer from part (a), students should not try to solve the problem using a method that does not follow from part (a).
- Teachers should emphasize that in justifications or explanations students need to correctly show that the necessary conditions have been satisfied. Merely stating the name of a theorem is usually not an acceptable explanation.
- Students who started taking derivatives of e^{-x^2} quickly found themselves having difficulty. Teachers should remind students that there are other, simpler methods for generating Taylor series centered at 0.
- Appropriate use of algebra is very important. Each step of a solution should follow the previous step via correct procedures.
- Teachers should give students more practice in dealing with subtleties of finding general terms and writing summation notation. Students should understand the differences between $\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{k!}$ and $\sum_{k=1}^{\infty}\frac{(-1)^k x^{2k}}{k!}$, between $\frac{(-1)^k x^{2k}}{k!}$ and $\frac{(-1)^{k+1} x^{2k}}{k!}$, and between $\frac{\left(-x^2\right)^k}{k!}$, $\frac{-x^{2k}}{k!}$, $\frac{(-x)^{2k}}{k!}$, and $\frac{\left(-x^k\right)^2}{k!}$.
- Correct mathematical notation is particularly important when dealing with infinite series.
- Students need additional practice in generating and using Maclaurin series (Taylor series centered at $x = 0$) for many of the elementary functions, including e^x, $\sin x$, $\cos x$, and $\frac{1}{1-x}$.
- Mathematical communication is very important. Students should communicate their understanding of the question and their solution in a clear, precise manner. They should use grammatically correct English, citing results of their earlier work when appropriate.

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