Question 3

Distance from the river's edge (feet)	0	8	14	22	24
Depth of the water (feet)	0	7	8	2	0

A scientist measures the depth of the Doe River at Picnic Point. The river is 24 feet wide at this location. The measurements are taken in a straight line perpendicular to the edge of the river. The data are shown in the table above. The velocity of the water at Picnic Point, in feet per minute, is modeled by $v(t) = 16 + 2\sin\left(\sqrt{t+10}\right)$ for $0 \le t \le 120$ minutes.

(a) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the area of the cross section of the river at Picnic Point, in square feet. Show the computations that lead to your answer.

(b) The volumetric flow at a location along the river is the product of the cross-sectional area and the velocity of the water at that location. Use your approximation from part (a) to estimate the average value of the volumetric flow at Picnic Point, in cubic feet per minute, from $t = 0$ to $t = 120$ minutes.

(c) The scientist proposes the function f, given by $f(x) = 8\sin\left(\frac{\pi x}{24}\right)$, as a model for the depth of the water, in feet, at Picnic Point x feet from the river's edge. Find the area of the cross section of the river at Picnic Point based on this model.

(d) Recall that the volumetric flow is the product of the cross-sectional area and the velocity of the water at a location. To prevent flooding, water must be diverted if the average value of the volumetric flow at Picnic Point exceeds 2100 cubic feet per minute for a 20-minute period. Using your answer from part (c), find the average value of the volumetric flow during the time interval $40 \le t \le 60$ minutes. Does this value indicate that the water must be diverted?

(a) $\frac{(0+7)}{2}\cdot 8 + \frac{(7+8)}{2}\cdot 6 + \frac{(8+2)}{2}\cdot 8 + \frac{(2+0)}{2}\cdot 2$
$= 115 \text{ ft}^2$

1 : trapezoidal approximation

(b) $\frac{1}{120}\int_0^{120} 115v(t)\,dt$
$= 1807.169 \text{ or } 1807.170 \text{ ft}^3/\text{min}$

$3 : \begin{cases} 1 : \text{limits and average value constant} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(c) $\int_0^{24} 8\sin\left(\frac{\pi x}{24}\right)dx = 122.230 \text{ or } 122.231 \text{ ft}^2$

$2 : \begin{cases} 1 : \text{integra1} \\ 1 : \text{answer} \end{cases}$

(d) Let C be the cross-sectional area approximation from part (c). The average volumetric flow is
$\frac{1}{20}\int_{40}^{60} C\cdot v(t)\,dt = 2181.912 \text{ or } 2181.913 \text{ ft}^3/\text{min}.$

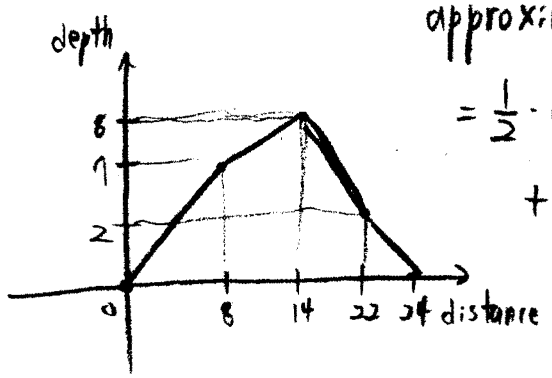
Yes, water must be diverted since the average volumetric flow for this 20-minute period exceeds 2100 ft^3/min.

$3 : \begin{cases} 1 : \text{volumetric flow integral} \\ 1 : \text{average volumetric flow} \\ 1 : \text{answer with reason} \end{cases}$

Form B
AB/BC 3
3A1

Distance from the river's edge (feet)	0	8	14	22	24
Depth of the water (feet)	0	7	8	2	0

Work for problem 3(a)



approximation of area using trapezoidal sum

$$= \frac{1}{2}\cdot(8)\cdot 7 + \frac{1}{2}\underbrace{(7+8)}_{=15}\cdot\underbrace{(14-8)}_{=6}$$

$$+ \frac{1}{2}\cdot\underbrace{(8+2)}_{=10}\cdot\underbrace{(22-14)}_{=8} + \frac{1}{2}\cdot 2\cdot\underbrace{(24-22)}_{=2}$$

$$= 4\cdot 7 + 3\cdot 15 + 5\cdot 8 + 2$$

$$= 28 + 45 + 40 + 2$$

$$= 73 + 42 = 115$$

$\underline{115\,(ft)^2}$ //

Work for problem 3(b)

Average value of volumetric flow at Picnic Point

$$= \frac{1}{120-0}\left(\int_0^{120} V(t)\,dt\right)\cdot 115 = \frac{115}{120}\int_0^{120}\left(16 + 2\sin\left(\sqrt{t+10}\right)\right)dt$$

$$= \underline{1809.16491\,(ft)^3/min}$$ //

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Continue problem 3 on page 9.

 3A2

Work for problem 3(c)

$$\text{Area} = \int_0^{24} 8\sin\left(\frac{\pi x}{24}\right) dx$$

$\cos\pi = -1$

$\cos(0) = 1$

$$= \left[-\frac{24}{\pi}\cdot 8\cos\left(\frac{\pi x}{24}\right)\right]_0^{24} = -\frac{24\cdot 8}{\pi}\cos(\pi) + \frac{24\cdot 8}{\pi}\cos(0)$$

$$= -\frac{24\cdot 8}{\pi}\cdot(-1) + \frac{24\cdot 8}{\pi}\cdot(1)$$

$$= 2\cdot\frac{24\cdot 8}{\pi} = \underline{122.23099 (ft)^2}$$

Work for problem 3(d)

Average value of volumetric flow during $40 \le t \le 60$

$$= \frac{1}{60-40}\left(\int_{40}^{60} \left(16 + 2\sin\left(\sqrt{t+10}\right)\right) dt\right)\cdot(122.23099)$$

$$= \frac{122.23099}{20}\cdot\int_{40}^{60} \left(16 + 2\sin\left(\sqrt{t+10}\right)\right) dt$$

$$= \underline{2181.91253\ (ft)^3/min} > 2100\ ft)^3/min$$

Thus, water must be diverted.

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

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Form B AB/BC 3

3B1

Distance from the river's edge (feet)	0	8	14	22	24
Depth of the water (feet)	0	7	8	2	0

Work for problem 3(a)

24ft

Trapezoidal sum = $\frac{\text{Right sum} + \text{Left sum}}{2}$

Right sum = $(2 \cdot 0) + (8 \cdot 2) + (6 \cdot 8) + (8 \cdot 7)$

$= 120$

Left sum = $(8 \cdot 0) + (6 \cdot 7) + (8 \cdot 8) + (2 \cdot 2)$

$= 110$

Trapezoidal sum = $\frac{120+110}{2} = \boxed{115 \text{ square feet}}$

Work for problem 3(b)

Flow = $115 \cdot v(t)$

$= 115 \cdot v(120)$

$= 115 \cdot 14.1627263406$

$= \boxed{1628.714 \text{ cubic feet per minute}}$

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Continue problem 3 on page 9.

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Work for problem 3(c)

(122.2309963)

$$\int_0^{24} f(x)\,dx = \boxed{122.231 \text{ square feet}}$$

Work for problem 3(d)

$\frac{1}{20}\int$ Volumetric flow > 2100 ft^3/min $\Rightarrow$ must be diverted.

$$\frac{1}{60-40}\int_{40}^{60} (122.231)(V(t))\,dt$$

$$= \frac{1}{20}(43638.25299)$$

$= \boxed{2181.913 \text{ ft}^3/\text{min.} \Rightarrow \text{Yes, this value indicates that the water must be diverted.}}$

END OF PART A OF SECTION II

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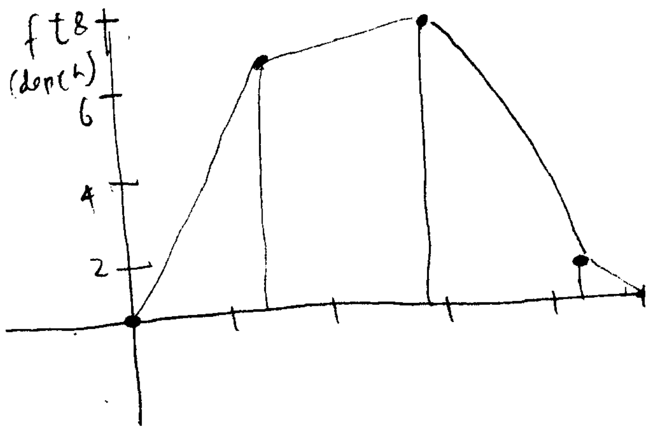
Form B
AB/BC 3
3C1

Distance from the river's edge (feet)	0	8	14	22	24
Depth of the water (feet)	0	7	8	2	0

Work for problem 3(a)



$$\frac{1}{2}(b_1+b_2)h + \frac{1}{2}(b_2+b_3)h + \frac{1}{2}(b_3+b_4)h + \frac{1}{2}(b_4+b_5)h.$$

$$= \frac{1}{2}(0+7)8 + \frac{1}{2}(7+8)6 + \frac{1}{2}(8+2)\times 8 + \frac{1}{2}(2+0)\times 2$$

$$= 115\ ft^2$$

$ft^2 \times ft/m$

Work for problem 3(b)

$$Avg = \frac{\text{approx Area} \times V(0) + \text{approx Area} \times V(120)}{\cancel{120}}$$

$$= \frac{115 \times (16+2\sin(\sqrt{0+10})) + 115\,(16+2\sin(\sqrt{120+10}))}{\cancel{120}}$$

$$= \cancel{28.9}\ ft^3/minute$$

3463

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Continue problem 3 on page 9.

Work for problem 3(c)

$$\text{area} = \int_0^{24} f(x)$$

$$\int_0^{24} 8\sin\left(\frac{\pi}{24}x\right)dx = -8 \times \frac{24}{\pi} \times \cos\left(\frac{\pi}{24}x\right)\Big|_0^{24}$$

$$= 122.231\ ft^2$$

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Work for problem 3(d)

$$\text{Area} \times V(40) + \text{Area} \times V(60)$$

$$= 122.231 \times (16 + 2\sin\sqrt{40+10}) + 122.231 \times (16 + 2\sin\sqrt{60+10})$$

$$= 4273.87\ ft^3/min$$

$\therefore$ must be diverted.

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END OF PART A OF SECTION II

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Question 3

Sample: 3A
Score: 9

The student earned all 9 points.

Sample: 3B
Score: 6

The student earned 6 points: 1 point in part (a), no points in part (b), 2 points in part (c), and 3 points in part (d). The student presents correct work in parts (a), (c), and (d). In part (b) the student does not produce an integral, which is needed in order to find the average value of volumetric flow.

Sample: 3C
Score: 3

The student earned 3 points: 1 point in part (a), no points in part (b), 2 points in part (c), and no points in part (d). The student presents correct work in parts (a) and (c). In part (b) the student does not produce an integral to find the average value and thus did not earn any points. In part (d) the student also does not produce an integral and did not earn any points. Although the student's statement that the water "must be diverted" is true, the student does not present enough correct calculus work leading up to the answer to earn the answer point.