AP® CALCULUS AB
2008 SCORING GUIDELINES

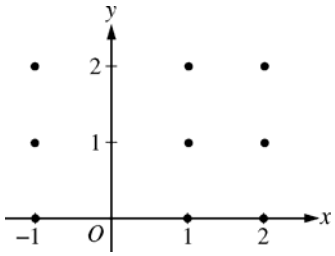
Question 5

Consider the differential equation $\frac{dy}{dx} = \frac{y-1}{x^2}$, where $x \neq 0$.

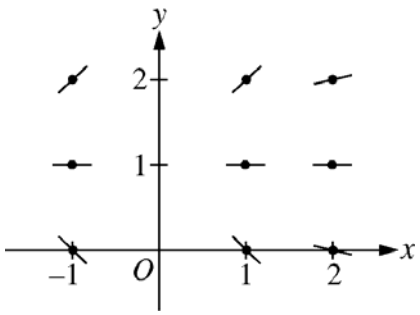
(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.
(Note: Use the axes provided in the exam booklet.)

(b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(2) = 0$.

(c) For the particular solution $y = f(x)$ described in part (b), find $\lim_{x\to\infty} f(x)$.



(a)



2 : { 1 : zero slopes; 1 : all other slopes }

(b) $\frac{1}{y-1}\,dy = \frac{1}{x^2}\,dx$

$\ln|y-1| = -\frac{1}{x} + C$

$|y-1| = e^{-\frac{1}{x}+C}$

$|y-1| = e^C e^{-\frac{1}{x}}$

$y - 1 = ke^{-\frac{1}{x}}$, where $k = \pm e^C$

$-1 = ke^{-\frac{1}{2}}$

$k = -e^{\frac{1}{2}}$

$f(x) = 1 - e^{\left(\frac{1}{2}-\frac{1}{x}\right)}$, $x > 0$

6 : { 1 : separates variables; 2 : antidifferentiates; 1 : includes constant of integration; 1 : uses initial condition; 1 : solves for y }

Note: max $3/6$ [1-2-0-0-0] if no constant of integration

Note: $0/6$ if no separation of variables

(c) $\lim_{x\to\infty} 1 - e^{\left(\frac{1}{2}-\frac{1}{x}\right)} = 1 - \sqrt{e}$

1 : limit

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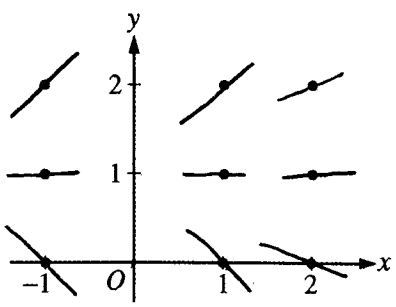
Work for problem 5(a)



$\frac{dy}{dx} = \frac{y-1}{x^2}$

$\frac{dy}{dx}$

$(-1,0): -1$
$(-1,1): 0$
$(-1,2): 1$
$(1,0): -1$
$(1,1): 0$
$(1,2): 1$
$(2,0): -\frac{1}{4}$
$(2,1): 0$
$(2,2): \frac{1}{4}$

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Continue problem 5 on page 13.

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5A2

Work for problem 5(b)

$\frac{dy}{dx} = \frac{y-1}{x^2}$

$\int \frac{dy}{y-1} = \int \frac{1}{x^2}\,dx$

$\int \frac{1}{y-1}\,dy = \int x^{-2}\,dx$

$\ln|y-1| = -x^{-1} + C$

$e^{\ln|y-1|} = e^{-x^{-1}+C}$

$y-1 = Ce^{-x^{-1}}$

$y = Ce^{-x^{-1}} + 1$

$0 = Ce^{-2^{-1}} + 1$

$0 = Ce^{-\frac{1}{2}} + 1$

$-1 = Ce^{-1/2}$

$-1 = \frac{C}{\sqrt{e}}$

$-\sqrt{e} = C$

$\therefore y = -\sqrt{e}\,e^{-x^{-1}} + 1$

Work for problem 5(c)

$\lim_{x\to\infty} -\sqrt{e}\,e^{-x^{-1}} + 1 = \lim_{x\to\infty} \frac{-\sqrt{e}}{e^{\frac{1}{x}}} + 1$

$\lim_{x\to\infty} \frac{-\sqrt{e}}{e^{\frac{1}{\infty}}} + 1$ ($\frac{1}{\infty} \to 0$)

$\lim_{x\to\infty} \frac{-\sqrt{e}}{1} + 1$

$\therefore \lim_{x\to\infty} f(x) = -\sqrt{e} + 1$

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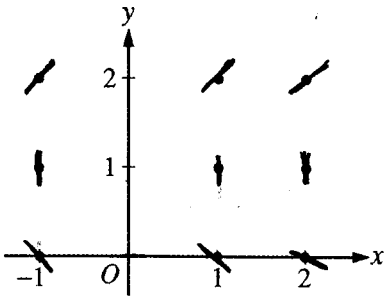
5B1

Work for problem 5(a)

$\frac{dy}{dx} = \frac{y-1}{x2}$



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5B2

Work for problem 5(b)

$$\frac{dy}{dx} = \frac{y-1}{x^2}$$

$$\int \frac{1}{y-1}\,dy = \int \frac{1}{x^2}\,dx$$

$$\ln|y-1| = -x^{-1} + C$$

$$\ln|0-1| = -(2)^{-1} + C$$

$$0 = -\frac{1}{2} + C$$

$$C = \frac{1}{2}$$

$$y = \frac{1}{2}(x-2)$$

Work for problem 5(c)

$$\lim_{x\to\infty} f(x) = \frac{1}{2}$$

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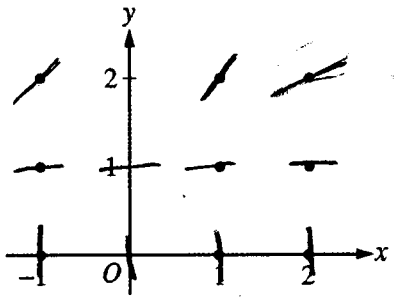
5C1

Work for problem 5(a)



see graph

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Work for problem 5(b)

$\frac{dy}{dx} = \frac{y-1}{x^2}$

$\int \frac{dy}{y-1} = \int \frac{dx}{x^2} \quad \ln(y-1) = \ln x^2 + C$

$y-1 = e^{\ln(x^2)+C}$

$y = Ce^{\ln x^2} + 1$

$0 = Ce^{\ln 4} + 1$

$Ce^{\ln 4} = -1$

$C = -\frac{1}{e^{\ln 4}}$

$y = -\frac{1}{e^{\ln 4}} e^{\ln x^2} + 1$

Work for problem 5(c)

this will keep getting bigger

$\lim_{x \to \infty} -\frac{e^{\ln x^2}}{e^{\ln 4}} + 1 = -\infty$

$\frac{-\infty}{+\infty} + 1$

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AP® CALCULUS AB
2008 SCORING COMMENTARY

Question 5

Overview

This problem presented students with a separable differential equation. In part (a) they were asked to sketch its slope field at nine sample points. Part (b) asked for the solution to the differential equation with a given initial condition. The solution involved selection of the portion of $\ln|y-1|$ that includes the initial condition. Part (c) asked for the limit of the solution from part (b) as $x \to \infty$.

Sample: 5A
Score: 9

The student earned all 9 points.

Sample: 5B
Score: 6

The student earned 6 points: 1 point in part (a), 5 points in part (b), and no points in part (c). In part (a) the student did not earn the first point but earned the second point. In part (b) the student earned the point for separation of variables, both points for antidifferentiating, and the points for the constant of integration and use of the initial condition. However, the student presents the equation for the tangent line at the initial condition point rather than solving for the function that satisfies the differential equation.

Sample: 5C
Score: 4

The student earned 4 points: 1 point in part (a), 3 points in part (b), and no points in part (c). In part (a) the student earned the first point. Since the slopes on the *x*-axis are not correct, the student did not earn the second point. The student also has slope segments drawn on the *y*-axis. In part (b) the student earned points for the separation of variables, the constant of integration, and use of the initial condition. The student did not earn any points for antidifferentiation. Since the antiderivative with respect to y does not include absolute values, the student is not able to correctly solve for y.

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