Question 2

A storm washed away sand from a beach, causing the edge of the water to get closer to a nearby road. The rate at which the distance between the road and the edge of the water was changing during the storm is modeled by $f(t) = \sqrt{t} + \cos t - 3$ meters per hour, t hours after the storm began. The edge of the water was 35 meters from the road when the storm began, and the storm lasted 5 hours. The derivative of $f(t)$ is $f'(t) = \frac{1}{2\sqrt{t}} - \sin t$.

(a) What was the distance between the road and the edge of the water at the end of the storm?

(b) Using correct units, interpret the value $f'(4) = 1.007$ in terms of the distance between the road and the edge of the water.

(c) At what time during the 5 hours of the storm was the distance between the road and the edge of the water decreasing most rapidly? Justify your answer.

(d) After the storm, a machine pumped sand back onto the beach so that the distance between the road and the edge of the water was growing at a rate of $g(p)$ meters per day, where p is the number of days since pumping began. Write an equation involving an integral expression whose solution would give the number of days that sand must be pumped to restore the original distance between the road and the edge of the water.

(a) $35 + \int_0^5 f(t)\,dt = 26.494$ or 26.495 meters	$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$
(b) Four hours after the storm began, the rate of change of the distance between the road and the edge of the water is increasing at a rate of $1.007\ \text{meters}/\text{hours}^2$.	$2: \begin{cases} 1: \text{interpretation of } f'(4) \\ 1: \text{units} \end{cases}$
(c) $f'(t) = 0$ when $t = 0.66187$ and $t = 2.84038$ The minimum of f for $0 \le t \le 5$ may occur at 0, 0.66187, 2.84038, or 5. $f(0) = -2$ $f(0.66187) = -1.39760$ $f(2.84038) = -2.26963$ $f(5) = -0.48027$ The distance between the road and the edge of the water was decreasing most rapidly at time $t = 2.840$ hours after the storm began.	$3: \begin{cases} 1: \text{considers } f'(t) = 0 \\ 1: \text{answer} \\ 1: \text{justification} \end{cases}$
(d) $-\int_0^5 f(t)\,dt = \int_0^x g(p)\,dp$	$2: \begin{cases} 1: \text{integral of } g \\ 1: \text{answer} \end{cases}$

Work for problem 2(a)

$F(0) = 35$

$35 + \int_0^5 F(t)\,dt \approx 35 - 8.505 \approx 26.495$ m

Work for problem 2(b)

$F'(4) = 1.007$

At 4 hours into the thunderstorm, the rate at which the distance between the road and the edge of the water was changing is increasing by 1.007 m/h².

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Work for problem 2(c)

$f'(t) = \frac{1}{2\sqrt{t}} - \sin t = 0$

$f'(0.662) = 0$ $\quad$ + | − $\quad$ .662 $\quad$ possible min

$f'(2.840) = 0$ $\quad$ − | + $\quad$ 2.840

$f(0) = -2$

$f(2.840) = -2.270$

$f(5) = -0.480$

Decreasing most rapidly at $t = 2.840$

Work for problem 2(d)

$\int_0^5 f(t)\,dt$ distance grown.. ≈ -8.505

$-8.505 + \int_0^t g(p)\,dp = 0$

$\int_0^t g(p)\,dp = 8.505$ m

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Work for problem 2(a)

Let $F(t)$ be the antiderivative of $f(x)$.

$\rightarrow F(t) = \frac{2}{3}t^{\frac{3}{2}} + \sin t - 3t + C$.

Since $F(0) = 35$, $C = 35$.

$\cancel{F(5) = \frac{2}{3} \times 5^{\frac{3}{2}} + \sin(5) - 3 \times 5 + C = 35}$.

$\cancel{C = 43.505}$.

$F(0) = \cancel{C = \boxed{43.505\,m}}$

$F(5) = \frac{2}{3} \times 5^{\frac{3}{2}} + \sin(5) - 3 \times 5 + 35$

$= \boxed{26.495\,m}$

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Work for problem 2(b)

$f(t)$ indicates the rate at which the distance between the road and the edge of the water was changing.

Therefore, $f'(t)$ indicates the rate at which the changing rate of the distance changes.

$f'(4) = 1.007$ means that the rate at which the ~~changing~~ changing rate of the distance between the road and the edge of the water is ~~1.007 meters per~~ $1.007\, m/hr^2$ when the storm lasted for 4 hours.

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Work for problem 2(c)

The distance between the road and the edge of the water decreases most rapidly. $\Leftrightarrow$ $f(t)$ is minimum.

$f(t)$ minimum ~~when~~ at the endpoint of $[0,5]$ or at the point at which $f'(t)=0$. ~~and $f''(t)>0$~~.

$f(0)=-3$; $f(5)=-0.480$.

$f'(t)=\frac{1}{2\sqrt{t}}-\sin t=0 \rightarrow t=0.662, 2.84_{cal}$

$f(0.662)=-1.372$.

$f(2.84)=-2.270$.

$\therefore$ minimum at $t=0$ (just when the storm started)

Work for problem 2(d)

The distance that needs to be restored is $35-26.495=8.505$m.

$\rightarrow \boxed{\int_0^x g(p)\,dp=8.505}$

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Work for problem 2(a)

$\cancel{\frac{dr}{dt}}$ $f(t) = \sqrt{t} + \cos t - 3$

$f'(t) = \frac{1}{2\sqrt{t}} - \sin t$ $\boxed{d = 35 \quad t = 0}$ $0 \le t \le 5$

(a) $d(5) = ?$ $\int_0^5 f(t)\,dt = d(5) - d(0)$

$= -8.50536.$

$\therefore d(5) = d(0) - 8.505$

$= \boxed{26.495\text{ m } (3.d.p)}$

Work for problem 2(b)

$f'(4) = 1.007$

$f'(4)$ means that during the fourth hour of the storm, the rate of change of 'the rate of change between the road and the edge of water' was 1.007. i.e.

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Work for problem 2(c)

d = distance between road and water

d decreasing most rapidly

$= \frac{d}{dt}$ ~~negative~~ max^{m} value at $\frac{d}{dt}(d) = f'(t)$

$= f''(t) = 0$.

$\therefore t = 4.68775$ or 42.4106

but $0 \le t \le 5$

$\therefore t = 4.688$ (3.d.p.) hour.

$=$ 4 hours 41 min (nearest whole min) after storm starts.

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Work for problem 2(d)

(d) $g(p)$

Sand lost during storm $= \int_0^5 f(t)\,dt$. →

$\therefore \int_0^5 f(t)\,dt = \int_0^p g(p)\,dp$ for sand to be restored to initial condition.

~~Sand pump in~~ ~~$\int_0^p g(p)\,dp = \int_0^5 f(t)\,dt$~~

~~Let $S(p)$ = sand pumped in at time p.~~

~~$\therefore$ Solution: $S(p)$~~

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Question 2

Sample: 2A
Score: 9

The student earned all 9 points. Note that in part (d) the student's second line earned both points. The t variable that the student uses in the first integral was ignored. That t is in hours after the start of the storm, but the t variable in the student's second integral is in days.

Sample: 2B
Score: 6

The student earned 6 points: 2 points in part (a), 1 point in part (b), 1 point in part (c), and 2 points in part (d). In part (a) the student's work is correct. The student does not include a definite integral but earned the integral point for correct antidifferentiation, use of the initial condition, and evaluation at 5. In part (b) the student earned the units point. Since the response does not include the word "increasing," the interpretation point was not earned. In part (c) the student earned the first point for considering $f'(t) = 0.$ The student did not earn the answer point due to evaluation errors and was not eligible for the justification point. In part (d) the student's boxed equation earned both points.

Sample: 2C
Score: 3

The student earned 3 points: 2 points in part (a), no points in part (b), no points in part (c), and 1 point in part (d). In part (a) the student's work is correct. In part (b) the response does not include the word "increasing" or any units. In part (c) the student is seeking a maximum value rather than a minimum value. The student considers $f''(t) = 0$ instead of $f'(t) = 0.$ In part (d) the student earned the point for the integral of g in spite of using the same name for the upper limit of integration and the variable of integration. The answer point was not earned since the response lacks a negative sign in the integral equation.