AP® Calculus AB
2009 Scoring Guidelines
Form B

Question 1

At a certain height, a tree trunk has a circular cross section. The radius $R(t)$ of that cross section grows at a rate modeled by the function

$$\frac{dR}{dt} = \frac{1}{16}\left(3 + \sin\left(t^2\right)\right) \text{ centimeters per year}$$

for $0 \le t \le 3$, where time t is measured in years. At time $t = 0$, the radius is 6 centimeters. The area of the cross section at time t is denoted by $A(t)$.

(a) Write an expression, involving an integral, for the radius $R(t)$ for $0 \le t \le 3$. Use your expression to find $R(3)$.

(b) Find the rate at which the cross-sectional area $A(t)$ is increasing at time $t = 3$ years. Indicate units of measure.

(c) Evaluate $\int_0^3 A'(t)\,dt$. Using appropriate units, interpret the meaning of that integral in terms of cross-sectional area.

(a) $R(t) = 6 + \int_0^t \frac{1}{16}\left(3 + \sin\left(x^2\right)\right) dx$ $R(3) = 6.610$ or 6.611	$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{expression for } R(t) \\ 1 : R(3) \end{cases}$
(b) $A(t) = \pi\left(R(t)\right)^2$ $A'(t) = 2\pi R(t) R'(t)$ $A'(3) = 8.858\ \text{cm}^2/\text{year}$	$3 : \begin{cases} 1 : \text{expression for } A(t) \\ 1 : \text{expression for } A'(t) \\ 1 : \text{answer with units} \end{cases}$
(c) $\int_0^3 A'(t)\,dt = A(3) - A(0) = 24.200$ or 24.201 From time $t = 0$ to $t = 3$ years, the cross-sectional area grows by 24.201 square centimeters.	$3 : \begin{cases} 1 : \text{uses Fundamental Theorem of Calculus} \\ 1 : \text{value of } \int_0^3 A'(t)\,dt \\ 1 : \text{meaning of } \int_0^3 A'(t)\,dt \end{cases}$

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2009 SCORING GUIDELINES (Form B)

Question 2

A storm washed away sand from a beach, causing the edge of the water to get closer to a nearby road. The rate at which the distance between the road and the edge of the water was changing during the storm is modeled by $f(t) = \sqrt{t} + \cos t - 3$ meters per hour, t hours after the storm began. The edge of the water was 35 meters from the road when the storm began, and the storm lasted 5 hours. The derivative of $f(t)$ is $f'(t) = \frac{1}{2\sqrt{t}} - \sin t$.

(a) What was the distance between the road and the edge of the water at the end of the storm?

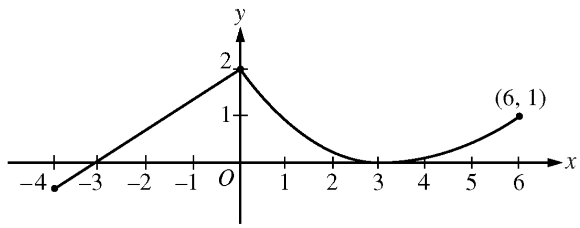
(b) Using correct units, interpret the value $f'(4) = 1.007$ in terms of the distance between the road and the edge of the water.

(c) At what time during the 5 hours of the storm was the distance between the road and the edge of the water decreasing most rapidly? Justify your answer.

(d) After the storm, a machine pumped sand back onto the beach so that the distance between the road and the edge of the water was growing at a rate of $g(p)$ meters per day, where p is the number of days since pumping began. Write an equation involving an integral expression whose solution would give the number of days that sand must be pumped to restore the original distance between the road and the edge of the water.

(a) $35 + \int_0^5 f(t)\,dt = 26.494$ or 26.495 meters	$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$
(b) Four hours after the storm began, the rate of change of the distance between the road and the edge of the water is increasing at a rate of 1.007 meters/hours^2.	$2: \begin{cases} 1: \text{interpretation of } f'(4) \\ 1: \text{units} \end{cases}$
(c) $f'(t) = 0$ when $t = 0.66187$ and $t = 2.84038$ The minimum of f for $0 \le t \le 5$ may occur at 0, 0.66187, 2.84038, or 5. $f(0) = -2$ $f(0.66187) = -1.39760$ $f(2.84038) = -2.26963$ $f(5) = -0.48027$ The distance between the road and the edge of the water was decreasing most rapidly at time $t = 2.840$ hours after the storm began.	$3: \begin{cases} 1: \text{considers } f'(t) = 0 \\ 1: \text{answer} \\ 1: \text{justification} \end{cases}$
(d) $-\int_0^5 f(t)\,dt = \int_0^x g(p)\,dp$	$2: \begin{cases} 1: \text{integral of } g \\ 1: \text{answer} \end{cases}$

Question 3



Graph of f

A continuous function f is defined on the closed interval $-4 \le x \le 6.$ The graph of f consists of a line segment and a curve that is tangent to the x-axis at $x = 3,$ as shown in the figure above. On the interval $0 < x < 6,$ the function f is twice differentiable, with $f''(x) > 0.$

(a) Is f differentiable at $x = 0$? Use the definition of the derivative with one-sided limits to justify your answer.

(b) For how many values of $a,$ $-4 \le a < 6,$ is the average rate of change of f on the interval $[a, 6]$ equal to 0 ? Give a reason for your answer.

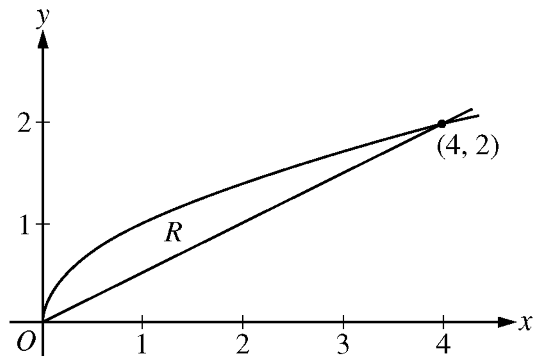
(c) Is there a value of $a,$ $-4 \le a < 6,$ for which the Mean Value Theorem, applied to the interval $[a, 6],$ guarantees a value $c,$ $a < c < 6,$ at which $f'(c) = \frac{1}{3}$? Justify your answer.

(d) The function g is defined by $g(x) = \int_0^x f(t)\,dt$ for $-4 \le x \le 6.$ On what intervals contained in $[-4, 6]$ is the graph of g concave up? Explain your reasoning.

(a) $\lim\limits_{h\to 0^-} \dfrac{f(h) - f(0)}{h} = \dfrac{2}{3}$ $\lim\limits_{h\to 0^+} \dfrac{f(h) - f(0)}{h} < 0$ Since the one-sided limits do not agree, f is not differentiable at $x = 0.$	$2 : \begin{cases} 1 : \text{sets up difference quotient at } x = 0 \\ 1 : \text{answer with justification} \end{cases}$
(b) $\dfrac{f(6) - f(a)}{6 - a} = 0$ when $f(a) = f(6).$ There are two values of a for which this is true.	$2 : \begin{cases} 1 : \text{expression for average rate of change} \\ 1 : \text{answer with reason} \end{cases}$
(c) Yes, $a = 3.$ The function f is differentiable on the interval $3 < x < 6$ and continuous on $3 \le x \le 6.$ Also, $\dfrac{f(6) - f(3)}{6 - 3} = \dfrac{1 - 0}{6 - 3} = \dfrac{1}{3}.$ By the Mean Value Theorem, there is a value $c,$ $3 < c < 6,$ such that $f'(c) = \dfrac{1}{3}.$	$2 : \begin{cases} 1 : \text{answers “yes” and identifies } a = 3 \\ 1 : \text{justification} \end{cases}$
(d) $g'(x) = f(x),$ $g''(x) = f'(x)$ $g''(x) > 0$ when $f'(x) > 0$ This is true for $-4 < x < 0$ and $3 < x < 6.$	$3 : \begin{cases} 1 : g'(x) = f(x) \\ 1 : \text{considers } g''(x) > 0 \\ 1 : \text{answer} \end{cases}$

Question 4

Let R be the region bounded by the graphs of $y = \sqrt{x}$ and $y = \frac{x}{2}$, as shown in the figure above.



(a) Find the area of R.

(b) The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are squares. Find the volume of this solid.

(c) Write, but do not evaluate, an integral expression for the volume of the solid generated when R is rotated about the horizontal line $y = 2$.

(a) Area $= \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right) dx = \frac{2}{3}x^{3/2} - \frac{x^2}{4}\Big|_{x=0}^{x=4} = \frac{4}{3}$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right)^2 dx = \int_0^4 \left(x - x^{3/2} + \frac{x^2}{4}\right) dx$

$= \frac{x^2}{2} - \frac{2x^{5/2}}{5} + \frac{x^3}{12}\Big|_{x=0}^{x=4} = \frac{8}{15}$

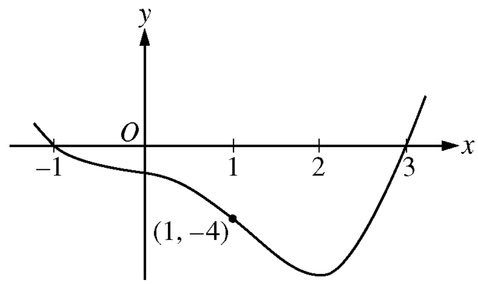
$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(c) Volume $= \pi \int_0^4 \left(\left(2 - \frac{x}{2}\right)^2 - \left(2 - \sqrt{x}\right)^2\right) dx$

$3 : \begin{cases} 1 : \text{limits and constant} \\ 2 : \text{integrand} \end{cases}$

Question 5

Let f be a twice-differentiable function defined on the interval $-1.2 < x < 3.2$ with $f(1) = 2$. The graph of f', the derivative of f, is shown above. The graph of f' crosses the x-axis at $x = -1$ and $x = 3$ and has a horizontal tangent at $x = 2$. Let g be the function given by $g(x) = e^{f(x)}$.



Graph of f'

(a) Write an equation for the line tangent to the graph of g at $x = 1$.

(b) For $-1.2 < x < 3.2$, find all values of x at which g has a local maximum. Justify your answer.

(c) The second derivative of g is $g''(x) = e^{f(x)}\left[(f'(x))^2 + f''(x)\right]$. Is $g''(-1)$ positive, negative, or zero? Justify your answer.

(d) Find the average rate of change of g', the derivative of g, over the interval $[1, 3]$.

(a) $g(1) = e^{f(1)} = e^2$

$g'(x) = e^{f(x)} f'(x)$, $g'(1) = e^{f(1)} f'(1) = -4e^2$

The tangent line is given by $y = e^2 - 4e^2(x - 1)$.

$3 : \begin{cases} 1 : g'(x) \\ 1 : g(1) \text{ and } g'(1) \\ 1 : \text{tangent line equation} \end{cases}$

(b) $g'(x) = e^{f(x)} f'(x)$

$e^{f(x)} > 0$ for all x

So, g' changes from positive to negative only when f' changes from positive to negative. This occurs at $x = -1$ only. Thus, g has a local maximum at $x = -1$.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

(c) $g''(-1) = e^{f(-1)}\left[(f'(-1))^2 + f''(-1)\right]$

$e^{f(-1)} > 0$ and $f'(-1) = 0$

Since f' is decreasing on a neighborhood of -1, $f''(-1) < 0$. Therefore, $g''(-1) < 0$.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

(d) $\dfrac{g'(3) - g'(1)}{3 - 1} = \dfrac{e^{f(3)} f'(3) - e^{f(1)} f'(1)}{2} = 2e^2$

$2 : \begin{cases} 1 : \text{difference quotient} \\ 1 : \text{answer} \end{cases}$

Question 6

t (seconds)	0	8	20	25	32	40
$v(t)$ (meters per second)	3	5	–10	–8	–4	7

The velocity of a particle moving along the x-axis is modeled by a differentiable function v, where the position x is measured in meters, and time t is measured in seconds. Selected values of $v(t)$ are given in the table above. The particle is at position $x = 7$ meters when $t = 0$ seconds.

(a) Estimate the acceleration of the particle at $t = 36$ seconds. Show the computations that lead to your answer. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_{20}^{40} v(t)\,dt$ in the context of this problem. Use a trapezoidal sum with the three subintervals indicated by the data in the table to approximate $\int_{20}^{40} v(t)\,dt$.

(c) For $0 \le t \le 40$, must the particle change direction in any of the subintervals indicated by the data in the table? If so, identify the subintervals and explain your reasoning. If not, explain why not.

(d) Suppose that the acceleration of the particle is positive for $0 < t < 8$ seconds. Explain why the position of the particle at $t = 8$ seconds must be greater than $x = 30$ meters.

(a) $a(36) = v'(36) \approx \dfrac{v(40) - v(32)}{40 - 32} = \dfrac{11}{8}$ meters/sec^2

1 : units in (a) and (b)

1 : answer

(b) $\int_{20}^{40} v(t)\,dt$ is the particle's change in position in meters from time $t = 20$ seconds to time $t = 40$ seconds.

$$\int_{20}^{40} v(t)\,dt \approx \frac{v(20) + v(25)}{2} \cdot 5 + \frac{v(25) + v(32)}{2} \cdot 7 + \frac{v(32) + v(40)}{2} \cdot 8$$
$$= -75 \text{ meters}$$

$3 : \begin{cases} 1 : \text{meaning of } \int_{20}^{40} v(t)\,dt \\ 2 : \text{trapezoidal approximation} \end{cases}$

(c) $v(8) > 0$ and $v(20) < 0$
$v(32) < 0$ and $v(40) > 0$
Therefore, the particle changes direction in the intervals $8 < t < 20$ and $32 < t < 40$.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{explanation} \end{cases}$

(d) Since $v'(t) = a(t) > 0$ for $0 < t < 8$, $v(t) \ge 3$ on this interval.

Therefore, $x(8) = x(0) + \int_0^8 v(t)\,dt \ge 7 + 8 \cdot 3 > 30.$

$2 : \begin{cases} 1 : v'(t) = a(t) \\ 1 : \text{explanation of } x(8) > 30 \end{cases}$