

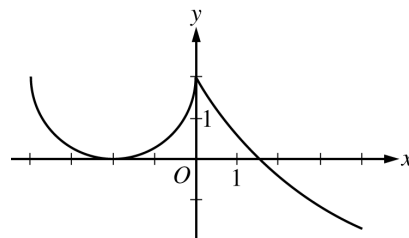
AP[®] CALCULUS AB
2009 SCORING GUIDELINES

Question 6

The derivative of a function f is defined by

$$f'(x) = \begin{cases} g(x) & \text{for } -4 \leq x \leq 0 \\ 5e^{-x/3} - 3 & \text{for } 0 < x \leq 4 \end{cases}$$

The graph of the continuous function f' , shown in the figure above, has x -intercepts at $x = -2$ and $x = 3\ln\left(\frac{5}{3}\right)$. The graph of g on $-4 \leq x \leq 0$ is a semicircle, and $f(0) = 5$.



Graph of f'

- (a) For $-4 < x < 4$, find all values of x at which the graph of f has a point of inflection. Justify your answer.
- (b) Find $f(-4)$ and $f(4)$.
- (c) For $-4 \leq x \leq 4$, find the value of x at which f has an absolute maximum. Justify your answer.

- (a) f' changes from decreasing to increasing at $x = -2$ and from increasing to decreasing at $x = 0$. Therefore, the graph of f has points of inflection at $x = -2$ and $x = 0$.

2 : $\begin{cases} 1 : \text{identifies } x = -2 \text{ or } x = 0 \\ 1 : \text{answer with justification} \end{cases}$

(b) $f(-4) = 5 + \int_0^{-4} g(x) dx$
 $= 5 - (8 - 2\pi) = 2\pi - 3$

$$f(4) = 5 + \int_0^4 (5e^{-x/3} - 3) dx$$

$$= 5 + \left(-15e^{-x/3} - 3x\right) \Big|_{x=0}^{x=4}$$

$$= 8 - 15e^{-4/3}$$

5 : $\begin{cases} 2 : f(-4) \\ 1 : \text{integral} \\ 1 : \text{value} \\ 3 : f(4) \\ 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{value} \end{cases}$

- (c) Since $f'(x) > 0$ on the intervals $-4 < x < -2$ and $-2 < x < 3\ln\left(\frac{5}{3}\right)$, f is increasing on the interval $-4 \leq x \leq 3\ln\left(\frac{5}{3}\right)$.

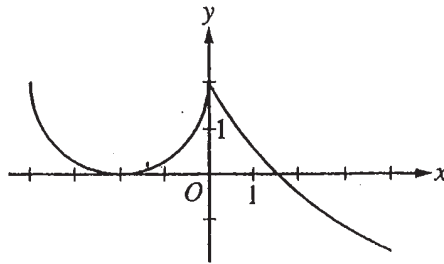
2 : $\begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

Since $f'(x) < 0$ on the interval $3\ln\left(\frac{5}{3}\right) < x < 4$, f is decreasing on the interval $3\ln\left(\frac{5}{3}\right) \leq x \leq 4$.

Therefore, f has an absolute maximum at $x = 3\ln\left(\frac{5}{3}\right)$.

6 6 6 6 6 6 6 6 6 6

NO CALCULATOR ALLOWED

Graph of f'

Work for problem 6(a)

The points of inflection are at

$$x = -2 \text{ and } x = 0$$

at $x = -2$ f' decreases then increases
 at $x = 0$ f' increases then decreases

Do not write beyond this border.

6 6 6 6 6 6 6 6 6 6

NO CALCULATOR ALLOWED

Work for problem 6(b)

$$\begin{aligned}
 f(-4) &= 5 - \int_{-4}^0 g(x) dx \\
 &= 5 - \left[4(x) - \frac{1}{2}\pi(x)^2 \right] \\
 &= 5 - [8 - 2\pi] = 2\pi - 3
 \end{aligned}$$

$$\begin{aligned}
 f(4) &= 5 + \int_0^4 (5e^{-x/3} - 3) dx \\
 &= 5 + [-3 \cdot 5e^{-x/3} - 3x]_0^4 \\
 &= 5 + [-15e^{-4/3} - 12 + 15] \\
 &= 5 - 15e^{-4/3} + 3 \\
 &= 8 - 15e^{-4/3}
 \end{aligned}$$

Do not write beyond this border.

Work for problem 6(c)

The absolute maximum is at $x = 3 \ln \frac{5}{3}$ since $f' > 0$ on the interval $(-4, -2)$ and $(-2, 3 \ln \frac{5}{3})$ which means f is increasing on those intervals also $f' < 0$ on the interval $(3 \ln \frac{5}{3}, 4]$ which means f is decreasing on this interval.

6

6

6

6

6

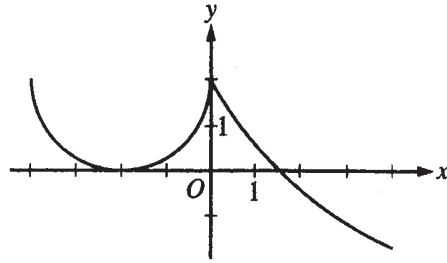
6

6

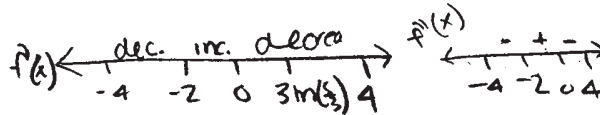
6

6

NO CALCULATOR ALLOWED

Graph of f'

Work for problem 6(a)



$f(x)$ has a point of inflection at $x = -2$ and $x = 0$
 because $f'(x)$ changes from \ominus to \oplus at $x = -2$
 and $f''(x)$ changes from \oplus to \ominus at $x = 0$

Do not write beyond this border.

Continue problem 6 on page 15.

6

6

6

6

6

NO CALCULATOR ALLOWED

Work for problem 6(b)

$$f(4) = \int_0^4 f'(x) dx$$

$$\frac{\pi r^2}{2} = \frac{\pi 4}{2} = 2\pi$$

$$2(4) = 8$$

$$\boxed{8 - 2\pi}$$

$$\begin{aligned} f(4) &= \int_0^4 f'(x) dx = \int_0^4 5e^{-x/3} - 3 dx \\ &= \left[-15e^{-x/3} - 3x \right]_0^4 \\ &= -15e^{-4/3} - 12 - (-15 - 0) \\ &= \boxed{-15e^{-4/3} + 3} \end{aligned}$$

Do not write beyond this border.

Work for problem 6(c)

$f(x)$ has an absolute maximum at $x = 3 \ln\left(\frac{5}{3}\right)$
because $f'(x) = 0$ at $x = 3 \ln\left(\frac{5}{3}\right)$ and changes
from \oplus to \ominus at $x = 3 \ln\left(\frac{5}{3}\right)$

GO ON TO THE NEXT PAGE.

6

6

6

6

6

6

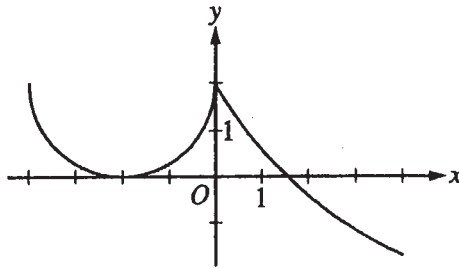
6

6

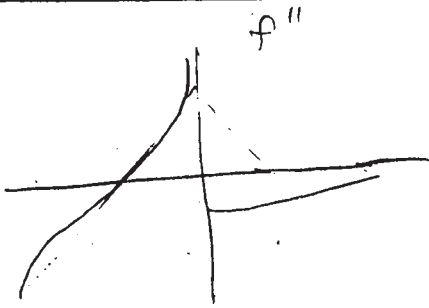
6

6

NO CALCULATOR ALLOWED

Graph of f'

Work for problem 6(a)



at -2 there is an inflection point.
 -2 is where $f' = 0$ and f'' changes
 sign.

Do not write beyond this border.

6

6

6

6

6

6

6

6

6

6

NO CALCULATOR ALLOWED

Work for problem 6(b)

$$f(-4) = \int_{-4}^0 g(x)$$

$$f(4) = \int_0^4 f'(x) dx = \int_0^4 5e^{-x/3} - 3 dx$$

$$= 15e^{-x/3} \Big|_0^4 - 3x \Big|_0^4$$

$$= 15e^{-4/3} - 15e^0 - 12$$

$$f(4) = -15e^{-4/3} + 3$$

$$f(4) = 8 - \frac{1}{2} \pi r^2 \quad r = 2$$

$$f(4) = 8 - 2\pi$$

Work for problem 6(c)

at $x = 1.5$, there is an absolute maximum
from $(-4, 1.5)$, $f(x)$ is increasing

at $f(1.5)$, $f'(x)$ sign changes, making $f(x)$ decrease

Do not write beyond this border.

AP[®] CALCULUS AB
2009 SCORING COMMENTARY

Question 6

Overview

In this problem a function f satisfies $f(0) = 5$ and has continuous first derivative for $-4 \leq x \leq 4$. The graph of f' was supplied. For $-4 \leq x \leq 0$, the graph of f' is a semicircle tangent to the x -axis at $x = -2$ and tangent to the y -axis at $y = 2$. For $0 < x \leq 4$, $f'(x) = 5e^{-x/3} - 3$. Part (a) asked for those values of x in the interval $-4 < x < 4$ at which the graph of f has a point of inflection; these correspond to points where the graph of f' changes from increasing to decreasing, or vice versa. In part (b) students had to use the given initial value for f and the appropriate piece of f' to find $f(-4)$ and $f(4)$. The former value required the evaluation of an integral using geometry, and the latter required the evaluation of an integral via an antiderivative. Part (c) asked for the value of x at which f attains its absolute maximum on the interval $[-4, 4]$. Using the derivative of f , students should have concluded that f is increasing on $\left[-4, 3\ln\left(\frac{5}{3}\right)\right]$ and decreasing on $\left[3\ln\left(\frac{5}{3}\right), 4\right]$, so that the maximum must occur at $x = 3\ln\left(\frac{5}{3}\right)$.

Sample: 6A

Score: 9

The student earned all 9 points.

Sample: 6B

Score: 6

The student earned 6 points: 2 points in part (a), 3 points in part (b), and 1 point in part (c). In part (a) the student's work is correct. In part (b) the student incorrectly uses $f'(x)$ as the integrand in the definite integral for $f(-4)$. However, the student earned the integral point by giving the correct geometric evaluation of the integral as $8 - 2\pi$. The student also earned the first 2 points for $f(4)$. The student did not earn either value point. In part (c) the student gives the correct absolute maximum. The justification point was not earned since the student does not provide a global argument.

Sample: 6C

Score: 4

The student earned 4 points: 1 point in part (a), 3 points in part (b), and no points in part (c). In part (a) the student earned the first point for one correct point of inflection but was not eligible for the justification point. In part (b) the student earned the integral point for $f(-4)$ as well as the first 2 points for $f(4)$. The student did not earn either value point. In part (c) the student's work is incorrect. The student estimates the x -intercept as 1.5 instead of using the information given in the question.