CollegeBoard AP

AP® Calculus AB
2009 Scoring Guidelines

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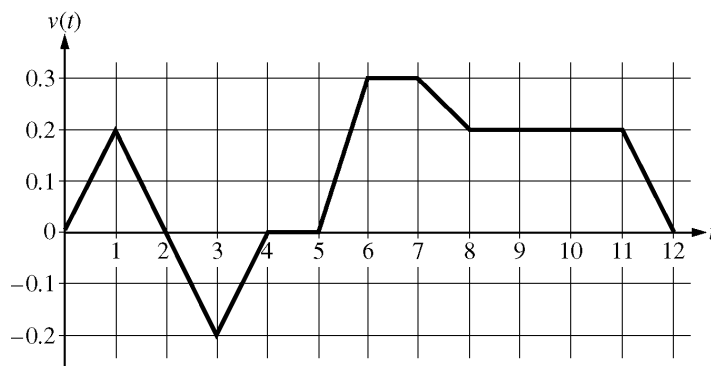
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Question 1

Caren rides her bicycle along a straight road from home to school, starting at home at time $t = 0$ minutes and arriving at school at time $t = 12$ minutes. During the time interval $0 \le t \le 12$ minutes, her velocity $v(t)$, in miles per minute, is modeled by the piecewise-linear function whose graph is shown above.

(a) Find the acceleration of Caren's bicycle at time $t = 7.5$ minutes. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_0^{12} |v(t)|\, dt$ in terms of Caren's trip. Find the value

of $\int_0^{12} |v(t)|\, dt$.

(c) Shortly after leaving home, Caren realizes she left her calculus homework at home, and she returns to get it. At what time does she turn around to go back home? Give a reason for your answer.

(d) Larry also rides his bicycle along a straight road from home to school in 12 minutes. His velocity is

modeled by the function w given by $w(t) = \dfrac{\pi}{15}\sin\left(\dfrac{\pi}{12}t\right)$, where $w(t)$ is in miles per minute for

$0 \le t \le 12$ minutes. Who lives closer to school: Caren or Larry? Show the work that leads to your answer.

(a) $a(7.5) = v'(7.5) = \dfrac{v(8) - v(7)}{8 - 7} = -0.1 \text{ miles}/\text{minute}^2$

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{units} \end{cases}$

(b) $\int_0^{12} |v(t)|\, dt$ is the total distance, in miles, that Caren rode

during the 12 minutes from $t = 0$ to $t = 12$.

$\int_0^{12} |v(t)|\, dt = \int_0^2 v(t)\, dt - \int_2^4 v(t)\, dt + \int_4^{12} v(t)\, dt$

$\qquad = 0.2 + 0.2 + 1.4 = 1.8 \text{ miles}$

$2 : \begin{cases} 1 : \text{meaning of integral} \\ 1 : \text{value of integral} \end{cases}$

(c) Caren turns around to go back home at time $t = 2$ minutes. This is the time at which her velocity changes from positive to negative.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

(d) $\int_0^{12} w(t)\, dt = 1.6$; Larry lives 1.6 miles from school.

$\int_0^{12} v(t)\, dt = 1.4$; Caren lives 1.4 miles from school.

Therefore, Caren lives closer to school.

$3 : \begin{cases} 2 : \text{Larry's distance from school} \\ \qquad 1 : \text{integral} \\ \qquad 1 : \text{value} \\ 1 : \text{Caren's distance from school} \\ \qquad \text{and conclusion} \end{cases}$

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2009 SCORING GUIDELINES

Question 2

The rate at which people enter an auditorium for a rock concert is modeled by the function R given by $R(t) = 1380t^2 - 675t^3$ for $0 \leq t \leq 2$ hours; $R(t)$ is measured in people per hour. No one is in the auditorium at time $t = 0$, when the doors open. The doors close and the concert begins at time $t = 2$.

(a) How many people are in the auditorium when the concert begins?

(b) Find the time when the rate at which people enter the auditorium is a maximum. Justify your answer.

(c) The total wait time for all the people in the auditorium is found by adding the time each person waits, starting at the time the person enters the auditorium and ending when the concert begins. The function w models the total wait time for all the people who enter the auditorium before time t. The derivative of w is given by $w'(t) = (2 - t)R(t)$. Find $w(2) - w(1)$, the total wait time for those who enter the auditorium after time $t = 1$.

(d) On average, how long does a person wait in the auditorium for the concert to begin? Consider all people who enter the auditorium after the doors open, and use the model for total wait time from part (c).

(a) $\int_0^2 R(t) \, dt = 980$ people

$$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$$

(b) $R'(t) = 0$ when $t = 0$ and $t = 1.36296$
The maximum rate may occur at 0, $a = 1.36296$, or 2.

$R(0) = 0$
$R(a) = 854.527$
$R(2) = 120$

The maximum rate occurs when $t = 1.362$ or 1.363.

$$3 : \begin{cases} 1 : \text{considers } R'(t) = 0 \\ 1 : \text{interior critical point} \\ 1 : \text{answer and justification} \end{cases}$$

(c) $w(2) - w(1) = \int_1^2 w'(t) \, dt = \int_1^2 (2 - t) R(t) \, dt = 387.5$
The total wait time for those who enter the auditorium after time $t = 1$ is 387.5 hours.

$$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$$

(d) $\frac{1}{980} w(2) = \frac{1}{980} \int_0^2 (2 - t) R(t) \, dt = 0.77551$
On average, a person waits 0.775 or 0.776 hour.

$$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$$

Mighty Cable Company manufactures cable that sells for $120 per meter. For a cable of fixed length, the cost of producing a portion of the cable varies with its distance from the beginning of the cable. Mighty reports that the cost to produce a portion of a cable that is x meters from the beginning of the cable is $6\sqrt{x}$ dollars per meter. (Note: Profit is defined to be the difference between the amount of money received by the company for selling the cable and the company's cost of producing the cable.)

(a) Find Mighty's profit on the sale of a 25-meter cable.

(b) Using correct units, explain the meaning of $\int_{25}^{30} 6\sqrt{x} \, dx$ in the context of this problem.

(c) Write an expression, involving an integral, that represents Mighty's profit on the sale of a cable that is k meters long.

(d) Find the maximum profit that Mighty could earn on the sale of one cable. Justify your answer.

(a) Profit $= 120 \cdot 25 - \int_{0}^{25} 6\sqrt{x} \, dx = 2500$ dollars

$$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$$

(b) $\int_{25}^{30} 6\sqrt{x} \, dx$ is the difference in cost, in dollars, of producing a cable of length 30 meters and a cable of length 25 meters.

1 : answer with units

(c) Profit $= 120k - \int_{0}^{k} 6\sqrt{x} \, dx$ dollars

$$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{expression} \end{cases}$$

(d) Let $P(k)$ be the profit for a cable of length k.
$P'(k) = 120 - 6\sqrt{k} = 0$ when $k = 400$.
This is the only critical point for P, and P' changes from positive to negative at $k = 400$.
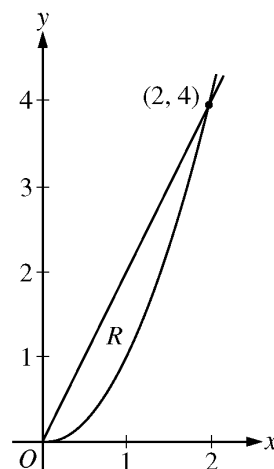Therefore, the maximum profit is $P(400) = 16,000$ dollars.

$$4 : \begin{cases} 1 : P'(k) = 0 \\ 1 : k = 400 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$$

Question 4

Let R be the region in the first quadrant enclosed by the graphs of $y = 2x$ and $y = x^2$, as shown in the figure above.

(a) Find the area of R.

(b) The region R is the base of a solid. For this solid, at each x the cross section perpendicular to the x-axis has area $A(x) = \sin\left(\dfrac{\pi}{2}x\right)$. Find the volume of the solid.

(c) Another solid has the same base R. For this solid, the cross sections perpendicular to the y-axis are squares. Write, but do not evaluate, an integral expression for the volume of the solid.

(a) Area $= \displaystyle\int_0^2 \left(2x - x^2\right) dx$

$= x^2 - \dfrac{1}{3}x^3 \Big|_{x=0}^{x=2}$

$= \dfrac{4}{3}$

$3: \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \displaystyle\int_0^2 \sin\left(\dfrac{\pi}{2}x\right) dx$

$= -\dfrac{2}{\pi}\cos\left(\dfrac{\pi}{2}x\right)\Big|_{x=0}^{x=2}$

$= \dfrac{4}{\pi}$

$3: \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(c) Volume $= \displaystyle\int_0^4 \left(\sqrt{y} - \dfrac{y}{2}\right)^2 dy$

$3: \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits} \end{cases}$

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Question 5

x	2	3	5	8	13
$f(x)$	1	4	–2	3	6

Let f be a function that is twice differentiable for all real numbers. The table above gives values of f for selected points in the closed interval $2 \leq x \leq 13$.

(a) Estimate $f'(4)$. Show the work that leads to your answer.

(b) Evaluate $\int_2^{13} (3 - 5f'(x))\, dx$. Show the work that leads to your answer.

(c) Use a left Riemann sum with subintervals indicated by the data in the table to approximate $\int_2^{13} f(x)\, dx$. Show the work that leads to your answer.

(d) Suppose $f'(5) = 3$ and $f''(x) < 0$ for all x in the closed interval $5 \leq x \leq 8$. Use the line tangent to the graph of f at $x = 5$ to show that $f(7) \leq 4$. Use the secant line for the graph of f on $5 \leq x \leq 8$ to show that $f(7) \geq \dfrac{4}{3}$.

(a) $f'(4) \approx \dfrac{f(5) - f(3)}{5 - 3} = -3$

$\qquad$ 1 : answer

(b) $\int_2^{13} (3 - 5f'(x))\, dx = \int_2^{13} 3\, dx - 5\int_2^{13} f'(x)\, dx$

$\qquad\qquad = 3(13 - 2) - 5(f(13) - f(2)) = 8$

$\qquad$ 2 : $\begin{cases} 1 : \text{uses Fundamental Theorem} \\ \qquad \text{of Calculus} \\ 1 : \text{answer} \end{cases}$

(c) $\int_2^{13} f(x)\, dx \approx f(2)(3 - 2) + f(3)(5 - 3)$

$\qquad\qquad + f(5)(8 - 5) + f(8)(13 - 8) = 18$

$\qquad$ 2 : $\begin{cases} 1 : \text{left Riemann sum} \\ 1 : \text{answer} \end{cases}$

(d) An equation for the tangent line is $y = -2 + 3(x - 5)$.

Since $f''(x) < 0$ for all x in the interval $5 \leq x \leq 8$, the line tangent to the graph of $y = f(x)$ at $x = 5$ lies above the graph for all x in the interval $5 < x \leq 8$.

Therefore, $f(7) \leq -2 + 3 \cdot 2 = 4$.

An equation for the secant line is $y = -2 + \dfrac{5}{3}(x - 5)$.

Since $f''(x) < 0$ for all x in the interval $5 \leq x \leq 8$, the secant line connecting $(5, f(5))$ and $(8, f(8))$ lies below the graph of $y = f(x)$ for all x in the interval $5 < x < 8$.

Therefore, $f(7) \geq -2 + \dfrac{5}{3} \cdot 2 = \dfrac{4}{3}$.

$\qquad$ 4 : $\begin{cases} 1 : \text{tangent line} \\ 1 : \text{shows } f(7) \leq 4 \\ 1 : \text{secant line} \\ 1 : \text{shows } f(7) \geq \dfrac{4}{3} \end{cases}$
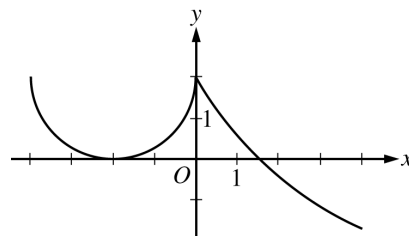
Question 6

The derivative of a function f is defined by

$$f'(x) = \begin{cases} g(x) & \text{for } -4 \le x \le 0 \\ 5e^{-x/3} - 3 & \text{for } 0 < x \le 4 \end{cases}.$$

The graph of the continuous function f', shown in the figure above, has

x-intercepts at $x = -2$ and $x = 3\ln\left(\frac{5}{3}\right)$. The graph of g on $-4 \le x \le 0$

is a semicircle, and $f(0) = 5$.

Graph of f'

(a) For $-4 < x < 4$, find all values of x at which the graph of f has a point of inflection. Justify your answer.

(b) Find $f(-4)$ and $f(4)$.

(c) For $-4 \le x \le 4$, find the value of x at which f has an absolute maximum. Justify your answer.

(a) f' changes from decreasing to increasing at $x = -2$ and from increasing to decreasing at $x = 0$. Therefore, the graph of f has points of inflection at $x = -2$ and $x = 0$.

$2 : \begin{cases} 1 : \text{identifies } x = -2 \text{ or } x = 0 \\ 1 : \text{answer with justification} \end{cases}$

(b) $f(-4) = 5 + \displaystyle\int_0^{-4} g(x)\, dx$

$\qquad = 5 - (8 - 2\pi) = 2\pi - 3$

$f(4) = 5 + \displaystyle\int_0^4 \left(5e^{-x/3} - 3\right) dx$

$\qquad = 5 + \left(-15e^{-x/3} - 3x\right)\Big|_{x=0}^{x=4}$

$\qquad = 8 - 15e^{-4/3}$

$5 : \begin{cases} 2 : f(-4) \\ \quad 1 : \text{integral} \\ \quad 1 : \text{value} \\ 3 : f(4) \\ \quad 1 : \text{integral} \\ \quad 1 : \text{antiderivative} \\ \quad 1 : \text{value} \end{cases}$

(c) Since $f'(x) > 0$ on the intervals $-4 < x < -2$ and

$-2 < x < 3\ln\left(\frac{5}{3}\right)$, f is increasing on the interval

$-4 \le x \le 3\ln\left(\frac{5}{3}\right)$.

Since $f'(x) < 0$ on the interval $3\ln\left(\frac{5}{3}\right) < x < 4$, f is

decreasing on the interval $3\ln\left(\frac{5}{3}\right) \le x \le 4$.

Therefore, f has an absolute maximum at $x = 3\ln\left(\frac{5}{3}\right)$.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$