Question 6

Two particles move along the x-axis. For $0 \le t \le 6$, the position of particle P at time t is given by $p(t) = 2\cos\left(\frac{\pi}{4}t\right)$, while the position of particle R at time t is given by $r(t) = t^3 - 6t^2 + 9t + 3$.

(a) For $0 \le t \le 6$, find all times t during which particle R is moving to the right.

(b) For $0 \le t \le 6$, find all times t during which the two particles travel in opposite directions.

(c) Find the acceleration of particle P at time $t = 3$. Is particle P speeding up, slowing down, or doing neither at time $t = 3$? Explain your reasoning.

(d) Write, but do not evaluate, an expression for the average distance between the two particles on the interval $1 \le t \le 3$.

(a) $r'(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3)$
$r'(t) = 0$ when $t = 1$ and $t = 3$
$r'(t) > 0$ for $0 < t < 1$ and $3 < t < 6$
$r'(t) < 0$ for $1 < t < 3$

Therefore R is moving to the right for $0 < t < 1$ and $3 < t < 6$.

$2 : \begin{cases} 1 : r'(t) \\ 1 : \text{answer} \end{cases}$

(b) $p'(t) = -2 \cdot \frac{\pi}{4}\sin\left(\frac{\pi}{4}t\right)$
$p'(t) = 0$ when $t = 0$ and $t = 4$

$p'(t) < 0$ for $0 < t < 4$
$p'(t) > 0$ for $4 < t < 6$

Therefore the particles travel in opposite directions for $0 < t < 1$ and $3 < t < 4$.

$3 : \begin{cases} 1 : p'(t) \\ 1 : \text{sign analysis for } p'(t) \\ 1 : \text{answer} \end{cases}$

(c) $p''(t) = -2 \cdot \frac{\pi}{4} \cdot \frac{\pi}{4}\cos\left(\frac{\pi}{4}t\right)$

$p''(3) = -2\left(\frac{\pi}{4}\right)^2 \cos\left(\frac{3\pi}{4}\right) = \frac{\pi^2}{8} \cdot \frac{\sqrt{2}}{2} > 0$

$p'(3) < 0$

Therefore particle P is slowing down at time $t = 3$.

$2 : \begin{cases} 1 : p''(3) \\ 1 : \text{answer with reason} \end{cases}$

(d) $\frac{1}{2}\int_1^3 \left| p(t) - r(t) \right| \, dt$

$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and constant} \end{cases}$

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Work for problem 6(a)

Partical R is moving to the right when $r'(t)>0$

$r'(t) = 3t^2 - 12t + 9 = 3\cdot(t^2-4t+3)$ ~~t~~ $r'(t)=0$

$t=1,\ t=3.$ $r'(t)>0$ for $t \in (0;1) \cup (3;6)$

Work for problem 6(b)

$p(t) = 2\cos\left(\frac{\pi}{4}t\right)$ $p'(t) = -2\sin\left(\frac{\pi}{4}t\right)\cdot\frac{\pi}{4} =$

$= -\frac{\pi}{2}\sin\left(\frac{\pi}{4}t\right)$ $p'(t)=0$ $t=4$ $p'(t)>0$ for $t\in(4;6)$

and $p'(t)<0$ for $t\in(0;4)$ In order for the particles to move in ~~the same~~ diffirent directions:

~~$p'(t)>0$ $t\in(4;6)$ $r'(t)>0$ $t\in(0;1)\cup(3;6)$~~

~~they move both p'~~

$$\left[\begin{array}{l}\begin{cases}p'(t)>0\\ r'(t)<0\end{cases}\\ \begin{cases}r'(t)>0\\ p'(t)<0\end{cases}\end{array}\right. \Leftrightarrow \left[\begin{array}{l}\begin{cases}\cancel{t\in(0;1)\cup(3;6)}\ t\in(4;6)\\ t\in(1;3)\end{cases}\\ \begin{cases}t\in(0;1)\cup(3;6)\\ t\in(0;4)\end{cases}\end{array}\right. \Leftrightarrow t\in(0;1)\cup(3;4)$$

Thus they move in diffirent directions for $t\in(0;1)\cup(3;4)$

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NO CALCULATOR ALLOWED

Work for problem 6(c)

$p(t) = 2\cos(\frac{\pi}{4}t)$ $\quad p'(t) = -2\sin(\frac{\pi}{4}t)\cdot\frac{\pi}{4} = -\frac{\pi}{2}\sin(\frac{\pi}{4}t)$

$p''(t) = (p'(t))' = -\frac{\pi^2}{8}\cos(\frac{\pi}{4}t)$ $\quad p''(3) = -\frac{\pi^2}{8}\cos(\frac{3}{4}\pi) = \frac{\pi^2}{8\sqrt{2}}$

~~as $p''(t) > 0$ the Particle P is speeding up at $t=3$.~~

$p'(3) = -\frac{\pi}{2}\sin(\frac{3}{4}\pi) = -\frac{\pi}{2\sqrt{2}}$

as $p'(3) < 0$ and $p''(3) > 0$ the particle is slowing down at $t=3$.

Work for problem 6(d)

$$\frac{1}{3-1}\int_1^3 |p(t) - r(t)|\,dt = \frac{1}{2}\int_1^3 |p(t) - r(t)|\,dt$$

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Work for problem 6(a) $r(t) > 0$

$r(t) = 3t^2 - 12t + 9 > 0$

$3(t^2 - 4t + 3) > 0$

$3(t-3)(t-1) > 0$

$t > 1,\ t > 3$

$\boxed{3 < t \leq 6}$

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Work for problem 6(b) $p'(t) = -\frac{\pi}{2}\sin\left(\frac{\pi}{4}t\right)$ $p'(t) < 0$ [0,4]

$p'(t) > 0$ [4,6]

whereas

$r'(t) < 0$ [0,3]

$r'(t) > 0$ (3,6]

Particles R and P are moving in opposite directions from time $3 < t \leq 4$

Continue problem 6 on page 15.

NO CALCULATOR ALLOWED

Work for problem 6(c)

$p''(t) = a(t)$

$p'(t) = -2\sin\left(\frac{\pi}{4}t\right)\cdot\frac{\pi}{4}$ $_2$

$p'(t) = -\frac{\pi}{2}\sin\left(\frac{\pi}{4}t\right)$

$p''(t) = -\frac{\pi}{2}\cos\left(\frac{\pi}{4}t\right)\frac{\pi}{4}$

$= -\frac{\pi^2}{8}\cos\left(\frac{\pi}{4}t\right)$

$p''(3) = -\frac{\pi^2}{8}\cos\left(\frac{3}{4}\pi\right)$

$\cos\left(\frac{3}{4}\pi\right) < 0 \therefore p''(3) > 0 \therefore$ speeding up.

Work for problem 6(d)

$\frac{1}{b-a}\int_a^b r(t) - p(t)\, dt$

$\frac{1}{2}\int_1^3 t^3 - 6t^2 + 9t + 3 - 2\cos\left(\frac{\pi}{4}t\right)\, dt$

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NO CALCULATOR ALLOWED

Work for problem 6(a)

$t \quad -3 \qquad -3t$

$t \quad -1 \qquad -t$

$t^2 \quad +3 \qquad -4t$

$r(t) = t^3 - 6t^2 + 9t + 3$

$r'(t) = 3t^2 - 12t + 9$

$r'(t) = 0$

$3t^2 - 12t + 9 = 0$

$t^2 - 4t + 3 = 0$

$(t-3)(t-1) = 0$

$t = 3$ or $t = 1$

Since R is moving to the right, $r(t) > 0$

$t = 3$ and $t = 1$

$r(1) = 1 - 6 + 9 + 3$

$= 7$

$r(3) = 27 - 6(9) + 9(3) + 3$

$= 27 - 54 + 27 + 3$

$= 3$

Work for problem 6(b)

$p(t) = 2\cos\left(\frac{\pi}{4} t\right)$

$p'(t) = -\cancel{2}\sin\left(\frac{\pi}{4}t\right)\left(\frac{\pi}{\cancel{4}2}\right)$

$= -\frac{\pi}{2}\sin\left(\frac{\pi}{4}t\right)$

When 2 particles travel in opposite directions,

$2\cos\left(\frac{\pi}{4}t\right) = t^3 - 6t^2 + 9t + 3$

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NO CALCULATOR ALLOWED

Work for problem 6(c)

$$P(t) = 2\cos\left(\tfrac{\pi}{4}t\right)$$

$$P'(t) = -\cancel{2}\sin\left(\tfrac{\pi}{4}t\right)\left(\tfrac{\pi}{\cancel{4}\,2}\right)$$

$$= -\tfrac{\pi}{2}\sin\left(\tfrac{\pi}{4}t\right)$$

$$A = P''(t) = -\tfrac{\pi}{2}\cos\left(\tfrac{\pi}{4}t\right)\left(\tfrac{\pi}{4}\right)$$

$$= -\tfrac{\pi}{2}\left(\tfrac{\pi}{4}\right)\cos\left(\tfrac{\pi}{4}t\right)$$

when $t = 3$

$$A = -\tfrac{\pi}{2}\left(\tfrac{\pi}{4}\right)\cos\left(\tfrac{3\pi}{4}\right)$$

$$= -\tfrac{\pi^2}{8}\cos\left(\tfrac{3\pi}{4}\right)$$

Particle P is slowing down because the accelaration of the particle is decreasing (<0).

Work for problem 6(d)

$$\text{Av. distance} = \frac{1}{3-1}\left(\left[2\cos\tfrac{\pi}{4}t\right]_1^3 + \left[t^3-6t^2+9t+3\right]_1^3\right)$$

Question 6

Sample: 6A
Score: 9

The student earned all 9 points.

Sample: 6B
Score: 6

The student earned 6 points: 1 point in part (a), 3 points in part (b), 1 point in part (c), and 1 point in part (d). In part (a) the student earned the point for $r'(t)$. Only one of the intervals is identified, so the second point was not earned. In part (b) the student's work is correct. The student's answer for when the two particles travel in opposite directions is consistent with the incorrect work in part (a); thus the point was earned. In part (c) the student earned the point for $p''(3)$, but the conclusion is not correct. In part (d) the student has the correct limits of integration and the correct constant factor but an incorrect integrand.

Sample: 6C
Score: 3

The student earned 3 points: 1 point in part (a), 1 point in part (b), 1 point in part (c), and no points in part (d). In part (a) the student earned the point for $r'(t)$. In part (b) the student earned the point for $p'(t)$. In part (c) the student earned the point for $p''(3)$. In part (d) the student does not provide an integral.