Question 3

Let R be the region in the first quadrant enclosed by the graphs of $f(x) = 8x^3$ and $g(x) = \sin(\pi x)$, as shown in the figure above.

(a) Write an equation for the line tangent to the graph of f at $x = \dfrac{1}{2}$.

(b) Find the area of R.

(c) Write, but do not evaluate, an integral expression for the volume of the solid generated when R is rotated about the horizontal line $y = 1$.
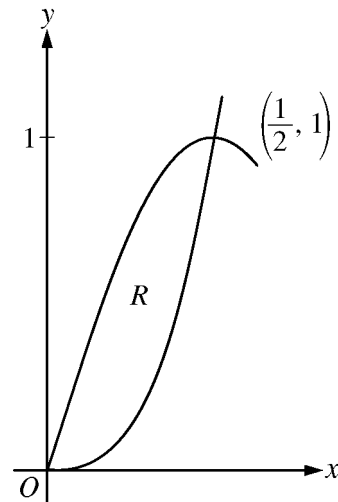
(a) $f\left(\dfrac{1}{2}\right) = 1$

$f'(x) = 24x^2$, so $f'\left(\dfrac{1}{2}\right) = 6$

An equation for the tangent line is $y = 1 + 6\left(x - \dfrac{1}{2}\right)$.

$$2 : \begin{cases} 1 : f'\left(\dfrac{1}{2}\right) \\ 1 : \text{answer} \end{cases}$$

(b) Area $= \displaystyle\int_0^{1/2} (g(x) - f(x))\, dx$

$\quad\quad = \displaystyle\int_0^{1/2} \left(\sin(\pi x) - 8x^3\right) dx$

$\quad\quad = \left[-\dfrac{1}{\pi}\cos(\pi x) - 2x^4 \right]_{x=0}^{x=1/2}$

$\quad\quad = -\dfrac{1}{8} + \dfrac{1}{\pi}$

$$4 : \begin{cases} 1 : \text{integrand} \\ 2 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$$

(c) $\pi \displaystyle\int_0^{1/2} \left((1 - f(x))^2 - (1 - g(x))^2 \right) dx$

$\quad = \pi \displaystyle\int_0^{1/2} \left(\left(1 - 8x^3\right)^2 - \left(1 - \sin(\pi x)\right)^2 \right) dx$

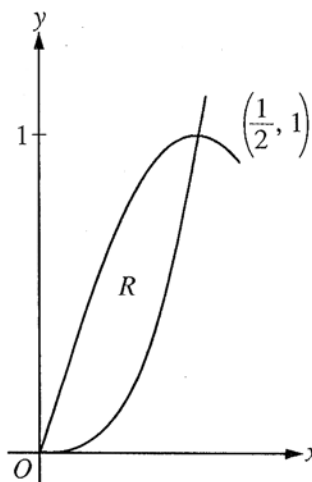$$3 : \begin{cases} 1 : \text{limits and constant} \\ 2 : \text{integrand} \end{cases}$$

CALCULUS AB
SECTION II, Part B
Time—60 minutes

Number of problems—4

No calculator is allowed for these problems.

$\left(\frac{1}{2}, 1\right)$

R

Work for problem 3(a)

$f(x) = 8x^3$

$f\left(\frac{1}{2}\right) = 8\left(\frac{1}{8}\right) = 1$ point of tangency : $\left(\frac{1}{2}, 1\right)$

$f'(x) = 24x^2$ $f'\left(\frac{1}{2}\right) = 24 \cdot \frac{1}{4} = 6$

$y = 6x + b$

$1 = 6\left(\frac{1}{2}\right) + b$

$1 = 3 + b$

$b = -2$

$y = 6x - 2$

The tangent line to the graph of f at $x = \frac{1}{2}$ is $y = 6x - 2$.

Do not write beyond this border.

-8-

Work for problem 3(b)

$$f(x) = g(x)$$
$$8x^3 = \sin(\pi x)$$
$$x = 0, \tfrac{1}{2}$$

let $u = \pi x$

$$R = \int_0^{\frac{1}{2}} \left[g(x) - f(x) \right] dx = \int_0^{\frac{1}{2}} \left[\sin(\pi x) - 8x^3 \right] dx = \frac{1}{\pi} \int_0^{\frac{1}{2}} \sin u \, du - \int_0^{\frac{1}{2}} 8x^3 dx$$

$$= \frac{1}{\pi}\left(-\cos(\pi x)\right)\Big]_0^{\frac{1}{2}} - 2x^4 \Big]_0^{\frac{1}{2}}$$

$$= \frac{1}{\pi}\left(-\cos\left(\frac{\pi}{2}\right) + \cos(0)\right) - 2\left(\frac{1}{2}\right)^4 + 0$$

$$= \frac{1}{\pi}(1) - \frac{2}{16}$$

$$= \frac{1}{\pi} - \frac{1}{8} = \boxed{\frac{8 - \pi}{8\pi}}$$

Work for problem 3(c)

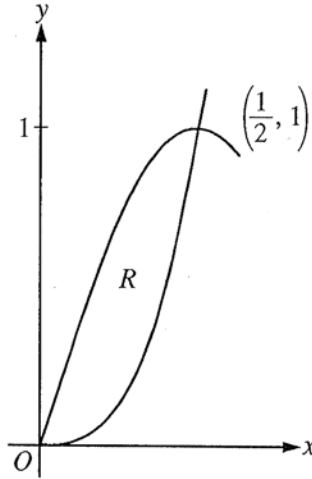$$\pi \int_0^{\frac{1}{2}} \left[\left(1 - f(x)\right)^2 - \left(1 - g(x)\right)^2 \right] dx$$

NO CALCULATOR ALLOWED

CALCULUS AB
SECTION II, Part B
Time—60 minutes

Number of problems—4

No calculator is allowed for these problems.

Work for problem 3(a)

$$f'(x) = 24x^2$$
$$24(1/2)^2 = 6$$

$$y - 1 = 6(x - 1/2)$$

-8-

Work for problem 3(b)

$$A = \int_{0}^{1/2} \left[\sin(\pi x) - 8x^3 \right] dx$$

Work for problem 3(c)

$$V = \pi \int_{0}^{1/2} \left[(1 - 8x^3)^2 - (1 - \sin(\pi x))^2 \right] dx$$
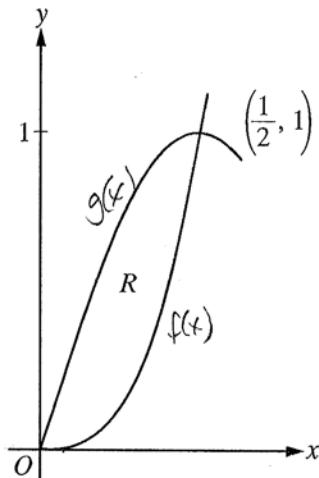
CALCULUS AB

SECTION II, Part B

Time—60 minutes

Number of problems—4

No calculator is allowed for these problems.

Work for problem 3(a)

$f(x) = 8x^3 \qquad f'(x) = 24x^2$

when $x = 1/2 \qquad f'(x) = 24\left(\frac{1}{4}\right) = 6$

$\boxed{y = 6x + b}$

-8-

Work for problem 3(b)

$$A = \int_0^{1/2} \sin(\pi x) - 8x^3 \; dx$$

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Do not write beyond this border.

Work for problem 3(c)

$$\pi \int_0^{1/2} \left(\sin(\pi x) - 8x^3 \right) dx = \text{Volume}$$

Question 3

Overview

This problem involved the graphs of functions $f(x) = 8x^3$ and $g(x) = \sin(\pi x)$ that enclose a region R in the first quadrant. A figure depicting R was supplied, with the label $\left(\frac{1}{2}, 1\right)$ at the point of intersection of the graphs of f and g. Part (a) asked for an equation of the line tangent to the graph of f at $x = \frac{1}{2}$. Part (b) asked for the area of R, which required students to set up and evaluate an appropriate definite integral. For part (c) students needed to provide an integral expression for the volume of the solid that is generated when R is rotated about the horizontal line $y = 1$.

Sample: 3A
Score: 9

The student earned all 9 points.

Sample: 3B
Score: 6

The student earned 6 points: 2 points in part (a), 1 point in part (b), and 3 points in part (c). In parts (a) and (c) the student's work is correct. In part (b) the student earned the integrand point only; the student does not find either antiderivative.

Sample: 3C
Score: 3

The student earned 3 points: 1 point in part (a), 1 point in part (b), and 1 point in part (c). In part (a) the student earned the point for $f'\left(\frac{1}{2}\right)$. The student does not find the y-intercept of the tangent line. In part (b) the student earned the integrand point. In part (c) the student earned the limits and constant point.