Question 5

At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that W will satisfy the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010.

(a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

(b) Find $\frac{d^2W}{dt^2}$ in terms of W. Use $\frac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \frac{1}{4}$.

(c) Find the particular solution $W = W(t)$ to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$.

(a) $\left.\frac{dW}{dt}\right|_{t=0} = \frac{1}{25}(W(0) - 300) = \frac{1}{25}(1400 - 300) = 44$

The tangent line is $y = 1400 + 44t$.

$W\left(\frac{1}{4}\right) \approx 1400 + 44\left(\frac{1}{4}\right) = 1411$ tons

$2 : \begin{cases} 1 : \frac{dW}{dt} \text{ at } t = 0 \\ 1 : \text{answer} \end{cases}$

(b) $\frac{d^2W}{dt^2} = \frac{1}{25}\frac{dW}{dt} = \frac{1}{625}(W - 300)$ and $W \ge 1400$

Therefore $\frac{d^2W}{dt^2} > 0$ on the interval $0 \le t \le \frac{1}{4}$.

The answer in part (a) is an underestimate.

$2 : \begin{cases} 1 : \frac{d^2W}{dt^2} \\ 1 : \text{answer with reason} \end{cases}$

(c) $\frac{dW}{dt} = \frac{1}{25}(W - 300)$

$\int \frac{1}{W - 300}\, dW = \int \frac{1}{25}\, dt$

$\ln|W - 300| = \frac{1}{25}t + C$

$\ln(1400 - 300) = \frac{1}{25}(0) + C \Rightarrow \ln(1100) = C$

$W - 300 = 1100e^{\frac{1}{25}t}$

$W(t) = 300 + 1100e^{\frac{1}{25}t}, \quad 0 \le t \le 20$

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } W \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

5A$_1$

Work for problem 5(a)

$25\overline{)1100}$ 44

at $t=0$, $W=1400$

so

$\left.\frac{dW}{dt}\right|_{t=0} = \frac{1}{25}(1100) = 44$ tons/year $= W'(0)$

$W(x+a) \approx W(x) + aW'(x)$

$W(0+\frac{1}{4}) \approx W(0) + \frac{1}{4}W'(0)$

$W(\frac{1}{4}) \approx 1400 + 11$

$W(\frac{1}{4}) \approx 1411$ tons

There will be about 1411 tons of landfill after 3 months.

Work for problem 5(b)

$\frac{dW}{dt} = \frac{1}{25}W - 12$

$\frac{d^2W}{dt^2} = \frac{1}{25}\frac{dW}{dt}$

$\boxed{\frac{d^2W}{dt} = \frac{1}{625}(W-300)}$

so $\frac{d^2W}{dt^2}$ is always positive b/c $W > 300$.

The answer in part a is an $\boxed{\text{underestimate}}$ because since $\frac{d^2W}{dt^2}$ is always positive for $t>0$, the graph of W is concave up, so the linearization of W is an underestimate.

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5A2

Work for problem 5(c)

$$\frac{dW}{dt} = \frac{1}{25}(W-300)$$

$$\int \frac{dW}{W-300} = \int \frac{1}{25}\, dt$$

$$e^{\ln|W-300|} = e^{\frac{1}{25}t + c}$$

can remove abs value bc $W > 300$. (incr fn)

$$\to |W-300| = C e^{\frac{1}{25}t}$$

Initial condition $W(0) = 1400$

$$1400 - 300 = C e^{0}$$

$$1100 = C$$

$$W - 300 = 1100 e^{\frac{1}{25}t}$$

$$\boxed{W = W(t) = 1100 e^{\frac{1}{25}t} + 300}$$

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NO CALCULATOR ALLOWED

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Work for problem 5(a)

$$\frac{dW}{dt} = \frac{1}{25}(1400-300) = \frac{1100}{25} \cdot \frac{1}{4} = \frac{1100}{100} = 11 + 1400 =$$

$$\boxed{1411 \text{ tons}}$$

Work for problem 5(b)

$$\frac{d^2W}{dt^2} = \frac{1}{25}W$$

$\boxed{\text{1411 tons is an underestimate}}$

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Work for problem 5(c)

$$\frac{dW}{dt} = \frac{1}{25}(W-300)$$

$$\int \frac{dW}{W-300} = \int \frac{1}{25}\,dt$$

$$e^{\ln|W-300|} = e^{\frac{1}{25}t + C}$$

$$W-300 = Ce^{\frac{1}{25}t}$$

$$W = Ce^{\frac{1}{25}t} + 300$$

$$1400 = Ce^{0} + 300$$

$$1400 = C + 300$$

$$C = 1100$$

$$\boxed{W = 1100e^{\frac{1}{25}t} + 300}$$

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5C1

NO CALCULATOR ALLOWED

Work for problem 5(a)

$\frac{dW}{(W-300)} = \frac{1}{25}\,dt$

$\ln(W-300) = \frac{1}{25}t$

$W-300 = e^{\frac{t}{25}}$

$W = e^{\frac{t}{25}} + 300 + 1400$

$W = e^{\frac{t}{25}} + 1700$

~~at $t=\frac{1}{4}$ the amount of waste is $e^{\frac{25}{4}} + 1700$~~

waste amount is $e^{\frac{25}{4}} + 1400$

Work for problem 5(b)

$\frac{d^2W}{dt^2} = \frac{1}{25}\left(\frac{dW}{dt} - 0\right)$

overestimate

$\frac{1}{25}W - \frac{300}{25}$

$\frac{dW}{dt} = \frac{1}{25}W - 12$

$\frac{d^2W}{dt^2} = \frac{1}{25}\frac{dW}{dt} - 0$

$\frac{d^2W}{dt^2}\ \frac{1}{25}\left(\frac{1}{25}(W-300)\right)$

$\frac{d^2W}{dt^2} = \frac{1}{625}(W-300)$

625

$\begin{array}{r} 25 \\ \times 25 \\ \hline 125 \\ 50 \end{array}$

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Work for problem 5(c)

$$\frac{1}{25}dt = \frac{dW}{(W-300)}$$

$$\frac{1}{25}t = \ln(W-300)$$

$$e^{\frac{t}{25}} = W-300$$

$$W = e^{\frac{t}{25}} + 300 + C$$

$$1400 = e^{\frac{0}{25}} + 300 + C$$

$$1400 = 301 + C$$

$$1099 = C$$

$$W = e^{\frac{t}{25}} + 301 + 1099$$

$$W = e^{\frac{t}{25}} + 1400$$

$$\begin{array}{r} 301 \\ \hline 00 \end{array}$$

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Question 5

Overview

The context of this problem was accumulating waste at a landfill. The landfill contained 1400 tons of waste at the beginning of 2010, and a function W modeling the total tons of waste in the landfill satisfies $\frac{dW}{dt} = \frac{1}{25}(W - 300)$, where t is measured in years since the start of 2010. Part (a) asked for an approximation to $W\left(\frac{1}{4}\right)$ using a tangent line approximation to the graph of W at $t = 0$. Part (b) asked for $\frac{d^2W}{dt^2}$ in terms of W, and students should have used a sign analysis of $\frac{d^2W}{dt^2}$ to determine whether the approximation in part (a) is an overestimate or an underestimate. Part (c) asked students to solve the initial value problem $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with $W(0) = 1400$ to find $W(t)$. Students should have used the method of separation of variables.

Sample: 5A
Score: 9

The student earned all 9 points.

Sample: 5B
Score: 6

The student earned 6 points: 1 point in part (a), no points in part (b), and 5 points in part (c). In part (a) the student earned the first point. The student has the correct answer of 1411 but links it incorrectly with equal signs to 44. As a result of this error, the second point was not earned. In part (b) the student's work is incorrect. In part (c) the student's work is correct.

Sample: 5C
Score: 3

The student earned 3 points: no points in part (a), 1 point in part (b), and 2 points in part (c). In part (a) the student's work is incorrect. In part (b) the student has a correct $\frac{d^2W}{dt^2}$. In part (c) the student earned the first 2 points for correct separation of variables and antidifferentiation. In part (c) absolute value symbols were not required for $\ln(W - 300)$ because $W > 1400$. The constant of integration appears late, so the student was not eligible for any additional points.