Question 6

Let f be a function defined by $f(x) = \begin{cases} 1 - 2\sin x & \text{for } x \le 0 \\ e^{-4x} & \text{for } x > 0. \end{cases}$

(a) Show that f is continuous at $x = 0$.

(b) For $x \ne 0$, express $f'(x)$ as a piecewise-defined function. Find the value of x for which $f'(x) = -3$.

(c) Find the average value of f on the interval $[-1, 1]$.

(a) $\lim\limits_{x \to 0^-} (1 - 2\sin x) = 1$

$\lim\limits_{x \to 0^+} e^{-4x} = 1$

$f(0) = 1$

So, $\lim\limits_{x \to 0} f(x) = f(0)$.

Therefore f is continuous at $x = 0$.

2 : analysis

(b) $f'(x) = \begin{cases} -2\cos x & \text{for } x < 0 \\ -4e^{-4x} & \text{for } x > 0 \end{cases}$

$-2\cos x \ne -3$ for all values of $x < 0$.

$-4e^{-4x} = -3$ when $x = -\frac{1}{4}\ln\left(\frac{3}{4}\right) > 0$.

Therefore $f'(x) = -3$ for $x = -\frac{1}{4}\ln\left(\frac{3}{4}\right)$.

$3 : \begin{cases} 2 : f'(x) \\ 1 : \text{value of } x \end{cases}$

(c)
$$\begin{aligned} \int_{-1}^{1} f(x)\,dx &= \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx \\ &= \int_{-1}^{0} (1 - 2\sin x)\,dx + \int_{0}^{1} e^{-4x}\,dx \\ &= \Big[x + 2\cos x\Big]_{x=-1}^{x=0} + \left[-\frac{1}{4}e^{-4x}\right]_{x=0}^{x=1} \\ &= (3 - 2\cos(-1)) + \left(-\frac{1}{4}e^{-4} + \frac{1}{4}\right) \end{aligned}$$

$$\begin{aligned} \text{Average value} &= \frac{1}{2}\int_{-1}^{1} f(x)\,dx \\ &= \frac{13}{8} - \cos(-1) - \frac{1}{8}e^{-4} \end{aligned}$$

$4 : \begin{cases} 1 : \int_{-1}^{0} (1 - 2\sin x)\,dx \text{ and } \int_{0}^{1} e^{-4x}\,dx \\ 2 : \text{antiderivatives} \\ 1 : \text{answer} \end{cases}$

6A,

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Work for problem 6(a)

To be continuous

i) $f(0) = 1$

ii) $\lim_{x\to 0^-} 1-2\sin x = \lim_{x\to 0^+} e^{-4x}$

$1 = 1 \quad \therefore \lim_{x\to 0^-} f = \lim_{x\to 0^+} f$

$\therefore \lim_{x\to 0} f = 1$

iii) $f(0) = \lim_{x\to 0} f = 1$

$\therefore$ f is continuous for all values of x.

Work for problem 6(b)

$$f'(x) = \begin{cases} -2\cos x, & x<0 \\ -4e^{-4x}, & x>0 \end{cases}$$

$f'(x) = -3$

-Since $-2\cos x$ oscillates between -2 and 2 there will be no such value in this function such that $f'(x) = -3$

But $f'(x) = -3$

$-4e^{-4x} = -3$

$e^{-4x} = 3/4$

$-4x = \ln(\frac{3}{4}) \quad \therefore x = \frac{-1}{4}\ln(\frac{3}{4}) \quad \therefore f'(\frac{-1}{4}\ln(\frac{3}{4})) = -3$

Continue problem 6 on page 15.

6A₂

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Work for problem 6(c)

$$f_{avg} = \frac{1}{1+1} \int_{-1}^{1} f(x)\,dx$$

$$= \frac{1}{2}\left[\int_{-1}^{0} 1 - 2\sin x\,dx + \int_{0}^{1} e^{-4x}\,dx \right]$$

$$= \frac{1}{2}\left[\left[x + 2\cos x\right]_{-1}^{0} + \left[\frac{-1}{4}e^{-4x}\right]_{0}^{1} \right]$$

$$\frac{1}{2}\left[2 + 1 - 2\cos(-1) + \frac{-e^{-4}}{4} + \frac{1}{4} \right]$$

$$= \frac{\left[3 - 2\cos(-1)\right] + \left[\frac{-1}{4e^4} + \frac{1}{4}\right]}{2}$$

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NO CALCULATOR ALLOWED

6B

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Work for problem 6(a)

$1-2\sin x = e^{-4x}$

$e^{-4(0)} = 1$ and $1-2\sin(0) = 1-0 = 1$

They're both $= 1$ @ $x=0$, therefore they're continuous.

Work for problem 6(b)

$$f'(x) = \begin{cases} -2\cos x & \text{for } x<0 \\ -4e^{-4x} & \text{for } x>0 \end{cases}$$

$-3 = -2\cos x$

$\frac{2}{3} = \cos x$

$\boxed{x = \cos^{-1}\frac{2}{3}}$

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Continue problem 6 on page 15.

NO CALCULATOR ALLOWED

6B2

Work for problem 6(c)

$$\frac{1}{2}\left(\int_{-1}^{0} 1-2\sin x\,dx + \int_{0}^{1} e^{-4x}\,dx\right)$$

$$\left(\left(x+2\cos x\right)\Big|_{-1}^{0}\right)+\left(-\frac{1}{4}e^{-4x}\Big|_{0}^{1}\right)$$

$$(0+2\cos 0)-(-1+2\cos -1)+\frac{-1}{4e^{4}}+\frac{1}{4}$$

$$2+1-2\cos(-1)+\frac{-1}{4e^{4}}+\frac{1}{4}$$

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6C,

Work for problem 6(a)

$1 - 2\sin x = e^{-4x}$

$1 - 2\sin(0) = e^{-4(0)}$

$1 = 1$

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Work for problem 6(b)

$$f'(x) = \begin{cases} -2\cos x \\ -4e^{-4x} \end{cases}$$

$x = -\frac{\ln\frac{3}{4}}{4}$

$-4e^{-4x} = -3$

$e^{-4x} = \frac{3}{4}$

$-4x = \ln\frac{3}{4}$

Continue problem 6 on page 15.

NO CALCULATOR ALLOWED

6C₂

Work for problem 6(c)

$$\frac{f(1)-f(-1)}{1+1} = \frac{e^{-4}-2\sin 1-1}{2}$$

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Question 6

Overview

This problem defined the function f using one expression for $x \le 0$ and a different expression for $x > 0$. Part (a) asked whether f is continuous at $x = 0$. Students needed to acknowledge that the left- and right-hand limits as $x \to 0$ and the value $f(0)$ all agree. Part (b) asked for a piecewise expression for $f'(x)$ and the value of x for which $f'(x) = -3$. This involves taking the symbolic derivatives of the branches of f and recognizing which piece produces a value of -3. Part (c) asked for the average value of f on the interval $[-1, 1]$. The required integral must be split at 0 to use the antiderivatives of the two branches of f.

Sample: 6A
Score: 9

The student earned all 9 points.

Sample: 6B
Score: 6

The student earned 6 points: 1 point in part (a), 2 points in part (b), and 3 points in part (c). In part (a) the student begins the analysis of continuity by looking at the functional values on each side of 0. The student does not use limits and does not consider $f(0) = 1$, thus earning only 1 of the possible 2 points. In part (b) the student presents a correct piecewise derivative, so the first 2 points were earned. The student's value of x is incorrect. In part (c) the student earned the first 3 points. The student does not multiply by $\frac{1}{2}$, so the answer point was not earned.

Sample: 6C
Score: 3

The student earned 3 points: 1 point in part (a), 2 points in part (b), and no points in part (c). In part (a) the student begins the analysis of continuity by looking at the functional values on each side of 0. The student does not use limits and does not consider $f(0) = 1$, thus earning only 1 of the possible 2 points. In part (b) the student does not give a correct piecewise presentation for $f'(x)$ and so earned 1 of the possible 2 points for $f'(x)$. The student finds the correct value of x and earned the third point in part (b). In part (c) the student's work is incorrect.