Question 4

Consider a differentiable function f having domain all positive real numbers, and for which it is known that $f'(x) = (4 - x)x^{-3}$ for $x > 0$.

(a) Find the x-coordinate of the critical point of f. Determine whether the point is a relative maximum, a relative minimum, or neither for the function f. Justify your answer.

(b) Find all intervals on which the graph of f is concave down. Justify your answer.

(c) Given that $f(1) = 2$, determine the function f.

(a) $f'(x) = 0$ at $x = 4$

$f'(x) > 0$ for $0 < x < 4$

$f'(x) < 0$ for $x > 4$

Therefore f has a relative maximum at $x = 4$.

$$3 : \begin{cases} 1 : x = 4 \\ 1 : \text{relative maximum} \\ 1 : \text{justification} \end{cases}$$

(b) $$\begin{aligned} f''(x) &= -x^{-3} + (4 - x)\left(-3x^{-4}\right) \\ &= -x^{-3} - 12x^{-4} + 3x^{-3} \\ &= 2x^{-4}(x - 6) \\ &= \frac{2(x - 6)}{x^4} \end{aligned}$$

$f''(x) < 0$ for $0 < x < 6$

The graph of f is concave down on the interval $0 < x < 6$.

$$3 : \begin{cases} 2 : f''(x) \\ 1 : \text{answer with justification} \end{cases}$$

(c) $$\begin{aligned} f(x) &= 2 + \int_1^x \left(4t^{-3} - t^{-2}\right) dt \\ &= 2 + \left[-2t^{-2} + t^{-1}\right]_{t=1}^{t=x} \\ &= 3 - 2x^{-2} + x^{-1} \end{aligned}$$

$$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$$

NO CALCULATOR ALLOWED

Work for problem 4(a)

$f(x) > 0$

for $f'(x) = 0$, $(4-x)x^{-3} = 0 \Rightarrow \frac{1}{x^3} \cdot (4-x) = 0$, $x \neq 0$

$\Rightarrow x = 4$ This is the x-coordinate of the critical point

Derivative test:

$f'(x)$ before $x=4$ is positive

$f'(x)$ after $x=4$ is negative

$f'(x) > 0$ for $0 < x < 4$

$f'(x) < 0$ for $4 < x < +\infty$

Thus the point with the x-coordinate $x=4$ is a relative maximum since at that point $f(x)$ goes from increasing to decreasing.

Work for problem 4(b)

$f'(x) = 4x^{-3} - x^{-2}$

f is concave down for $f''(x) = 0$

$f''(x) = -12x^{-4} + 2x^{-3} = 0$

$\Rightarrow -6x^{-4} + x^{-3} = 0$

$\Rightarrow x^{-3}\left(\frac{-6}{x} + 1\right) = 0$

$\Rightarrow \frac{1}{x^3}\left(-\frac{6}{x} + 1\right) = 0 \Rightarrow 1 - \frac{6}{x} = 0$

$\Rightarrow x = 6$

$f''(x) < 0$ for $0 < x < 6$

and $f''(x) > 0$ for $6 < x < +\infty$

Thus the interval on which f is concave down is $0 < x < 6$

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NO CALCULATOR ALLOWED

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Work for problem 4(c)

$$f(x) = \int_1^x (4u^{-3} - u^{-2})\,dx + 2$$

$$= \left[-2u^{-2} + u^{-1}\right]_1^x + 2$$

$$= -2x^{-2} + x^{-1} - [-2+1] + 2$$

$$= -2x^{-2} + x^{-1} + 2 - 1 + 2$$

$$= -2x^{-2} + x^{-1} + 3 .$$

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NO CALCULATOR ALLOWED

Work for problem 4(a)

$f'(x) = (4-x)x^{-3} = 0$

$4x^{-3} - x^{-2} = 0$

$x^{-2}\left(\frac{4}{x} - 1\right) = 0$

$x = 0, \quad \frac{4}{x} - 1 = 0$

$\frac{4}{x} = 1$

$x = 4$

interval	$x < 0$	$0 < x \le 4$	$x > 4$
sign $f'(x)$	$-$	$+$	$-$
Behavior	Decr	Incr	Decr

at $x = 0 \rightarrow$ relative minimum

at $x = 4 \rightarrow$ relative maximum

Work for problem 4(b)

$f''(x) = (4-x)(-3x^{-4}) + (x^{-3})(-1)$

$= -12x^{-4} + 3x^{-3} - x^{-3}$

$= -x^{-3}(12x^{-1} - 3 + 1) = 0$

$\frac{12}{x} = 2$

$x = 6$

Interval	$x < 6$	$x \ge 6$
sign $f''(x)$	$+$	$+$
Behavior	concave up	concave up

f is never concave down because $f''(x)$ is always positive

Continue problem 4 on page 11.

NO CALCULATOR ALLOWED

Work for problem 4(c)

$f(x) = \int 4x^{-3} - x^{-2} \, dx$

$= -2x^{-2} + x^{-1} + C$

$2 = -2(1)^{-2} + (1)^{-1} + C$

$2 = -2 + 1 + C$

$C = 3$

$\downarrow$

$f(x) = -2x^{-2} + x^{-1} + 3$

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Work for problem 4(a)

$f'(x) = 0$

$(4-x)x^{-3} = 0$

$x^{-3} = 0 \quad | \quad x = 4$

$x = 0$

$f'(4) = (4-4)(4)^{-3} = 0$

~~neither~~

we cant tell if this is a relative max or min, so its neither

Work for problem 4(b)

$u = 4-x \quad | \quad v = x^{-3}$

$u' = -1 \quad | \quad v' = -3x^{-4}$

$f''(x) = -x^{-3} + (-12x^{-4} + 3x^{-3})$

x	$(0,4)$	$(4,\infty)$
$f''(x)$	\	/

at $(0,4)$ its a concave down

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NO CALCULATOR ALLOWED

Work for problem 4(c)

$u = \cancel{4}-x$ $\quad dx = -du$

$du = -1\ dx$

$f(x) = \int (4-x)x^{-3}\ dx$

$= -\left[\frac{ux^{-2}}{-2} + C\right]$

$= \frac{(4-x)\ x^{-2}}{2} + C$

$f(1) = \frac{(4-1)\ (1)^{-2}}{2} + C$

$2 = \frac{3}{2} + C$

$\boxed{C = \frac{1}{2}}$

$f(x) = \frac{(4-x)x^{-2}}{2} + \frac{1}{2}$

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Question 4

Sample: 4A
Score: 9

The student earned all 9 points. In part (c) the student's differential does not match the variable in the integrand, but no points were deducted.

Sample: 4B
Score: 6

The student earned 6 points: 1 point in part (a), 2 points in part (b), and 3 points in part (c). The student was eligible for 1 of the 3 points in part (a), which was earned by showing that f has a relative maximum at $x = 4.$ In part (b) the student correctly computes $f''(x),$ so the first 2 points were earned. In part (c) the student's work is correct.

Sample: 4C
Score: 3

The student earned 3 points: no points in part (a), 2 points in part (b), and 1 point in part (c). In part (a) the student incorrectly declares that both $x = 0$ and $x = 4$ are x-coordinates of critical points. In part (b) the student correctly computes $f''(x),$ so the first 2 points were earned. The student's interval is incorrect, so the answer point was not earned. In part (c) the student presents a differential equation solution. For a correct solution, the first point was earned for a correct antiderivative, the second point for use of the initial condition, and the third point for the answer. This student's antiderivative is incorrect, so the first point was not earned. The initial condition is used correctly, so the second point was earned. Because the student's antidifferentiation is incorrect, the student was not eligible for the answer point.