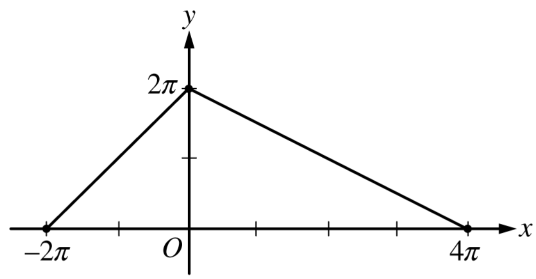
Question 6

Let g be the piecewise-linear function defined on $[-2\pi, 4\pi]$ whose graph is given above, and let $f(x) = g(x) - \cos\left(\frac{x}{2}\right)$.



Graph of g

(a) Find $\int_{-2\pi}^{4\pi} f(x)\,dx$. Show the computations that lead to your answer.

(b) Find all x-values in the open interval $(-2\pi, 4\pi)$ for which f has a critical point.

(c) Let $h(x) = \int_0^{3x} g(t)\,dt$. Find $h'\left(-\frac{\pi}{3}\right)$.

(a) $$\int_{-2\pi}^{4\pi} f(x)\,dx = \int_{-2\pi}^{4\pi} \left(g(x) - \cos\left(\frac{x}{2}\right)\right) dx$$
$$= 6\pi^2 - \left[2\sin\left(\frac{x}{2}\right)\right]_{x=-2\pi}^{x=4\pi}$$
$$= 6\pi^2$$

$2 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(b) $$f'(x) = g'(x) + \frac{1}{2}\sin\left(\frac{x}{2}\right) = \begin{cases} 1 + \frac{1}{2}\sin\left(\frac{x}{2}\right) & \text{for } -2\pi < x < 0 \\ -\frac{1}{2} + \frac{1}{2}\sin\left(\frac{x}{2}\right) & \text{for } 0 < x < 4\pi \end{cases}$$

$f'(x)$ does not exist at $x = 0$.
For $-2\pi < x < 0$, $f'(x) \neq 0$.
For $0 < x < 4\pi$, $f'(x) = 0$ when $x = \pi$.

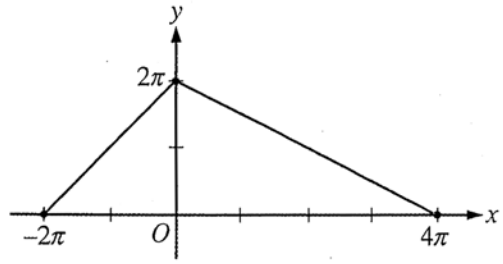
f has critical points at $x = 0$ and $x = \pi$.

$4 : \begin{cases} 1 : \frac{d}{dx}\left(\cos\left(\frac{x}{2}\right)\right) \\ 1 : g'(x) \\ 1 : x = 0 \\ 1 : x = \pi \end{cases}$

(c) $h'(x) = g(3x) \cdot 3$
$h'\left(-\frac{\pi}{3}\right) = 3g(-\pi) = 3\pi$

$3 : \begin{cases} 2 : h'(x) \\ 1 : \text{answer} \end{cases}$

NO CALCULATOR ALLOWED



Graph of g

Work for problem 6(a)

$$f(x) = g(x) - \cos\left(\frac{x}{2}\right)$$

$$\int_{-2\pi}^{4\pi} f(x)\,dx$$

$$= \int_{-2\pi}^{4\pi} g(x) - \cos\left(\frac{x}{2}\right)\,dx$$

$$= \int_{-2\pi}^{4\pi} g(x)\,dx - \int_{-2\pi}^{4\pi} \cos\left(\frac{x}{2}\right)dx$$

$$= (4\pi+2\pi)\cdot \cancel{2}\pi \cdot \frac{1}{\cancel{2}} - \int_{-2\pi}^{4\pi} \cos\left(\frac{x}{2}\right)dx$$

$$= 6\pi^2 - \left[2\sin\left(\frac{x}{2}\right)\right]_{-2\pi}^{4\pi}$$

$$= 6\pi^2 - \left[2\sin(2\pi) - 2\sin(-\pi)\right]$$

$$= 6\pi^2 - [0 - 0]$$

$$= 6\pi^2$$

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Continue problem 6 on page 15.

Work for problem 6(b)

$$f'(x) = g'(x) - \cos'\left(\frac{x}{2}\right)$$

$$= g'(x) + \frac{1}{2}\sin\left(\frac{x}{2}\right)$$

critical point: ① $f'(x)$ undefined at $x=0$

$f'(x) = 0$. $g'(x) = -\frac{1}{2}\sin\left(\frac{x}{2}\right)$

on $(-2\pi, 0)$ $\sin\left(\frac{x}{2}\right) = -2$. not exist.

on $(0, 4\pi)$ $\cancel{-\frac{1}{2}}\sin\left(\frac{x}{2}\right) = \frac{-2\pi}{4\pi} = \cancel{-\frac{1}{2}}$ $\sin\left(\frac{x}{2}\right) = 1$

$\frac{x}{2} = \frac{\pi}{2}, \frac{5}{2}\pi, \dots$

② $x \in (0, 4\pi)$ $\therefore x = \pi$

Therefore. $x = 0$, $x = \pi$. f has a critical point.

Work for problem 6(c)

$$h(x) = \int_0^{3x} g(t)\,dt$$

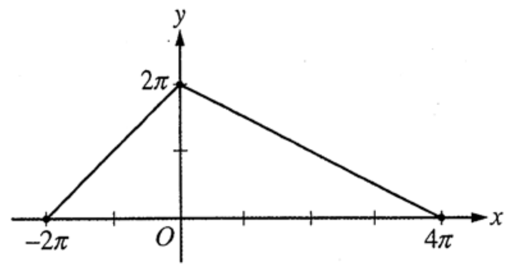
$$h'(x) = 3g(3x)$$

$$h'\left(-\frac{\pi}{3}\right) = 3 \cdot g(-\pi)$$

$$= 3 \cdot \pi$$

$$h'\left(-\frac{\pi}{3}\right) = 3\pi.$$

NO CALCULATOR ALLOWED



Graph of g

Work for problem 6(a)

$$\int_{-2\pi}^{4\pi} f(x)\,dx = \int_{-2\pi}^{4\pi} \left(g(x) - \cos\left(\tfrac{x}{2}\right)\right) dx$$

$$= \int_{-2\pi}^{4\pi} g(x)\,dx - \int_{-2\pi}^{4\pi} \cos\left(\tfrac{x}{2}\right) dx$$

$$= \int_{-2\pi}^{0} g(x)\,dx + \int_{0}^{4\pi} g(x)\,dx - 2\int_{-2\pi}^{4\pi} \tfrac{1}{2}\cos\left(\tfrac{x}{2}\right) dx$$

$$= \tfrac{1}{\cancel{2}}(\cancel{2}\pi)(2\pi) + \tfrac{1}{\cancel{2}}(4\pi)(\cancel{2}\pi) - \left[(2)\left[\sin \tfrac{x}{2}\Big|_{-2\pi}^{4\pi}\right.\right.$$

$$= 2\pi^2 + 4\pi^2 - \left[(2)\left[\sin 2\pi - \sin(-\pi)\right]\right]$$

$$= 6\pi^2 - \left[-2\sin(-\pi)\right]$$

$$= 6\pi^2 + 2\sin(-\pi) = 6\pi^2$$

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Continue problem 6 on page 15.

Work for problem 6(b)

$f'(x) = g'(x) + \frac{1}{2}\sin\left(\frac{x}{2}\right)$

$= g'(x) + \frac{1}{2}\sin\frac{x}{2}$

$f'(x) = 0 \implies g'(x) = -\frac{1}{2}\sin\frac{x}{2}$

$\therefore x = 4\pi$

$\therefore \boxed{x = 4\pi}$ is a critical point

because $g'(4\pi) = 0$

and $-\frac{1}{2}\sin\frac{4\pi}{2} = -\frac{1}{2}\sin 2\pi = 0$

$\underline{f'(x) = 0}$

Work for problem 6(c)

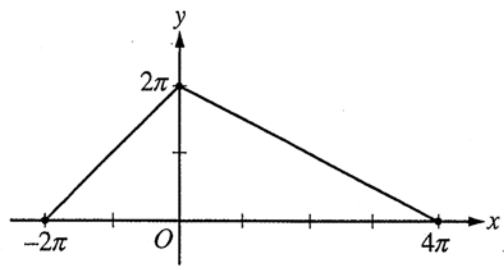
$h(x) = \int_0^{3x} g(t)\,dt$

$\therefore h'(x) = 3\,g(3x)$

$h'\left(-\frac{\pi}{3}\right) = 3\,g\left(3\left(\frac{-\pi}{3}\right)\right)$

$= 3\,g(-\pi) = 3(\pi) = 3\pi$

NO CALCULATOR ALLOWED

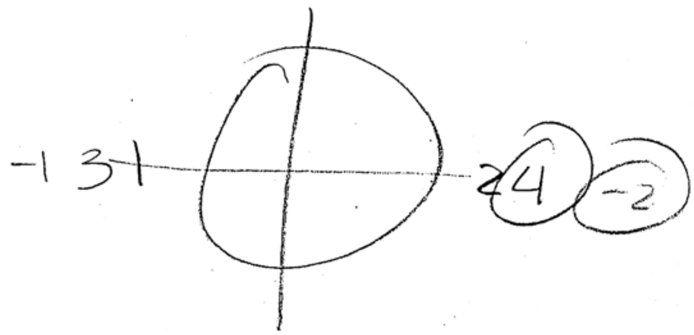


Graph of g

Work for problem 6(a)

$f(x) = g(x) - \cos\left(\frac{x}{2}\right)$

$A = \int_{-2\pi}^{4\pi} g(x) - \cos\frac{x}{2}\, dx$



$A = \int_{-2\pi}^{4\pi} g(x)\,dx + \int_{-2\pi}^{4\pi} -\cos\frac{x}{2}\, dx$

$= \left(\frac{1}{2}\, 2\pi(2\pi) + \frac{1}{2}\, 4\pi(2\pi)\right) +$

$= (2\pi^2 + 4\pi^2) +$

$= 6\pi^2 + \int_{-2\pi}^{4\pi} -\cos\frac{x}{2}\, dx$

$= \quad + \frac{1}{2}\sin\frac{x}{2}\Big|_{-2\pi}^{4\pi}$

$= \quad + \frac{1}{2}(0 - 0) = 6\pi^2$

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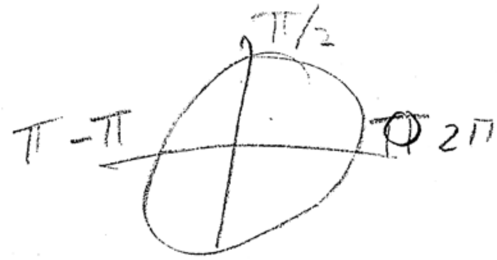
Continue problem 6 on page 15.

Work for problem 6(b)

$f(x)=g(x)-\cos\left(\frac{x}{2}\right)$

$0=g(x)-\cos\left(\frac{x}{2}\right)$

$\cos\left(\frac{x}{2}\right)=g(x)$



?

Work for problem 6(c)

$h(c)=\int_0^{3r} g(t)\,dt$

$h'(c)=g(3x)$

$h'\left(-\frac{\pi}{3}\right)=g\left(-\cancel{\frac{\pi}{3}(3)}\right)$

$=g(-\pi)=\pi$

$h'\left(-\frac{\pi}{3}\right)=\pi$

☆ fundamental theorem of calculus

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Question 6

Sample: 6A
Score: 9

The student earned all 9 points. In part (b) the student earned the $g'(x)$ point implicitly on the fifth and sixth lines of the solution. The student presents multiple solutions on the seventh line but then reports the desired ones.

Sample: 6B
Score: 6

The student earned 6 points: 2 points in part (a), 1 point in part (b), and 3 points in part (c). In parts (a) and (c) the student's work is correct. In part (b) the student earned the first point. The student does not present $g'(x)$ and does not identify the critical points.

Sample: 6
Score: 3

The student earned 3 points: 1 point in part (a), no points in part (b), and 2 points in part (c). In part (a) the student has an antidifferentiation error on the next to last line, so the first point was not earned. The student was eligible for the answer point and earned it. In part (c) the response is missing a factor of 3 and so earned only 1 of the 2 points for $h'(x)$. The student was eligible for the answer point and earned it.