2017

AP CollegeBoard

AP Calculus AB

Sample Student Responses and Scoring Commentary

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Question 4

(a) $H'(0) = -\frac{1}{4}(91 - 27) = -16$ $H(0) = 91$ An equation for the tangent line is $y = 91 - 16t$. The internal temperature of the potato at time $t = 3$ minutes is approximately $91 - 16 \cdot 3 = 43$ degrees Celsius.	$3 : \begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line} \\ 1 : \text{approximation} \end{cases}$
(b) $\frac{d^2H}{dt^2} = -\frac{1}{4}\frac{dH}{dt} = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)(H - 27) = \frac{1}{16}(H - 27)$ $H > 27$ for $t > 0 \Rightarrow \frac{d^2H}{dt^2} = \frac{1}{16}(H - 27) > 0$ for $t > 0$ Therefore, the graph of H is concave up for $t > 0$. Thus, the answer in part (a) is an underestimate.	1 : underestimate with reason
(c) $\frac{dG}{(G - 27)^{2/3}} = -dt$ $\int \frac{dG}{(G - 27)^{2/3}} = \int (-1)\, dt$ $3(G - 27)^{1/3} = -t + C$ $3(91 - 27)^{1/3} = 0 + C \Rightarrow C = 12$ $3(G - 27)^{1/3} = 12 - t$ $G(t) = 27 + \left(\frac{12 - t}{3}\right)^3$ for $0 \le t < 10$ The internal temperature of the potato at time $t = 3$ minutes is $27 + \left(\frac{12 - 3}{3}\right)^3 = 54$ degrees Celsius.	$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration and} \\ \quad \text{uses initial condition} \\ 1 : \text{equation involving } G \text{ and } t \\ 1 : G(t) \text{ and } G(3) \end{cases}$ Note: max $2/5$ [1-1-0-0-0] if no constant of integration Note: $0/5$ if no separation of variables

NO CALCULATOR ALLOWED

4A₁

4. At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

$\left.\frac{dH}{dt}\right|_{H=91} = -\frac{1}{4}(91-27) = \left(-\frac{1}{4}\right)(64) = -16$

$\begin{array}{r} 8\,11 \\ 27 \\ \hline 64 \end{array}$

$A(t) = -16(t-0) + 91$

$A(t) = -16t + 91$

$A(3) = -16(3) + 91$

$= -48 + 91$

$= \boxed{43°C \text{ at } t = 3 \text{ minutes}}$

$\begin{array}{r} 8\,11 \\ 91 \\ -48 \\ \hline 43 \end{array}$

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

$\frac{dH}{dt} = -\frac{1}{4}(H-27)$

$\frac{d^2H}{dt^2} = -\frac{1}{4}\left(\frac{dH}{dt}\right)$

$= -\frac{1}{4}\left(-\frac{1}{4}\right)(H-27)$

$\frac{d^2H}{dt^2} = \frac{1}{16}(H-27)$

$H > 27$ for all $t > 0$

$\frac{d^2H}{dt^2}$ is always positive, so part (a) is an

underestimate.

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NO CALCULATOR ALLOWED

4A₂

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

$$\frac{dG}{dt} = -(G-27)^{2/3}$$

$$\int \frac{dG}{(G-27)^{2/3}} = \int -1\,dt$$

$$\int (G-27)^{-2/3}\,dG = -t+C$$

$$3(G-27)^{1/3} = -t + C_1$$

$$(G-27)^{1/3} = -\frac{t}{3} + C_2$$

$$G-27 = \left(-\frac{t}{3} + C_2\right)^3$$

$$G = \left(-\frac{t}{3} + C_2\right)^3 + 27$$

at $(0, 91)$

$$91 = (0 + C_2)^3 + 27$$

$$91 = (C_2)^3 + 27$$

$$64 = (C_2)^3 \rightarrow C_2 = 4$$

91
27

$$\boxed{G(t) = \left(-\frac{t}{3} + 4\right)^3 + 27}$$

$$G(3) = \left(-\frac{3}{3} + 4\right)^3 + 27$$

$$= (3)^3 + 27$$

$$= 27 + 27$$

$$\boxed{G(3) = 54^\circ C}$$

4B1

4. At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

$\frac{dH}{dt} = -\frac{1}{4}(91-27)$

$\frac{dH}{dt} = -\frac{1}{4}(64)$

$\frac{dH}{dt} = -16$

equation of tangent line:

$y - 91 = -16t$

$y = -16t + 91$

approximation at time $t=3$:

$y = -16(3) + 91$

$y = -48 + 91$

$= 43°C$ at time $t=3$

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

$\frac{dH'}{dt} = -\frac{1}{4}(H-27)$

$\frac{dH}{dt} = -\frac{1}{4}H + \frac{27}{4}$

$\frac{d^2H}{dt^2} = -\frac{1}{4}$

underestmate because the value is less that the estimated value of $\frac{d^2H}{dt^2}$ at $t=3$

Do not write beyond this border.

Do not write beyond this border.

Continue question 4 on page 17.

4B$_2$

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

$$\frac{1}{-(G-27)^{2/3}}\, dG = 1\, dt$$

$$3(G-27)^{1/3} = t + C$$

$$3(91-27)^{1/3} = 0 + C$$

$$3(64)^{1/3} = C$$

$$3(4) = C$$

$$12 = C$$

$$3(G-27)^{1/3} = t + 12$$

$$(G-27)^{\cancel{1/3}} = \left(\frac{t+12}{3}\right)^3 + 27$$

$$G(t) = \left(\frac{t+12}{3}\right)^3 + 27$$

$$G(3) = \left(\frac{15}{3}\right)^3 + 27$$

$$G(3) = (5)^3 + 27$$

$$G(3) = 125 + 27$$

$$G(3) = 152$$

152°C at time $t = 3$

4C1

4. At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

$$\frac{dH}{dt} = -\frac{1}{4}(91-27) = -\frac{1}{4}(64) = -16$$

$$H - 91 = -16(t)$$

$$H = -16t + 91$$

$$H = -16(3) + 91$$

$$H = -48 + 91$$

$$H = 43°C$$

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

$$\frac{dH}{dt} = -\frac{1}{4}H + \frac{27}{4}$$

$$\frac{d^2H}{dt^2} = \frac{-1}{4}$$

This is an underestimate because $\frac{dH}{dt}$ + $\frac{d^2H}{dt^2}$ are both decreasing

Continue question 4 on page 17.

4C2

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

$$\frac{dG}{dt} = -(G-27)^{\frac{2}{3}}$$

$$\int \frac{dG}{dt} = \int -(G-27)^{\frac{2}{3}} dt$$

$$G(t) = -\int (G-27)^{\frac{2}{3}} dt \qquad u = G-27 \quad du = dt$$

$$G(t) = -\int u^{\frac{2}{3}} du$$

$$G(t) = -\frac{u^{\frac{5}{3}}}{\frac{5}{3}} \quad -\frac{3}{5}(G-27)^{\frac{5}{3}}$$

$$-\frac{3}{5}(91-27)^{\frac{5}{3}}$$

$$-\frac{3}{5}(64)^{\frac{5}{3}}$$

$64 \times 4 = 256$, $256 \times 4 = 1024$

Question 4

Overview

The context for this problem is the internal temperature of a boiled potato that is left to cool in a kitchen. Initially at time $t = 0,$ the potato's internal temperature is 91 degrees Celsius, and it is given that the internal temperature of the potato exceeds 27 degrees Celsius for all times $t > 0$. The internal temperature of the potato at time t minutes is modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27),$ where $H(t)$ is measured in degrees Celsius and $H(0) = 91.$ In part (a) students were asked for an equation of the line tangent to the graph of H at $t = 0,$ and to use this equation to approximate the internal temperature of the potato at time $t = 3.$ Using the initial value and the differential equation, students needed to find the slope of the tangent line to be $H'(0) = -\frac{1}{4}(91 - 27) = -16$ and report the equation of the tangent line to be $y = 91 - 16t.$ Students needed to find the approximate temperature of the potato at $t = 3$ to be $91 - 16 \cdot 3 = 43$ degrees Celsius. [LO 2.3B/EK 2.3B2] In part (b) students were asked to use $\frac{d^2H}{dt^2}$ to determine whether the approximation in part (a) is an underestimate or overestimate for the potato's internal temperature at time $t = 3.$ Students needed to use the given differential equation to calculate $\frac{d^2H}{dt^2} = -\frac{1}{4}\frac{dH}{dt} = \frac{1}{16}(H - 27).$ Then using the given information that the temperature always exceeds 27 degrees Celsius, students needed to conclude that $\frac{d^2H}{dt^2} > 0$ for all times t. Thus, the graph of H is concave up, and the line tangent to the graph of H at $t = 0$ lies below the graph of H (except at the point of tangency), so the approximation found in part (a) is an underestimate. [LO 2.1D/EK 2.1D1, LO 2.2A/EK 2.2A1] In part (c) an alternate model, G, is proposed for the internal temperature of the potato at times $t < 10.$ $G(t)$ is measured in degrees Celsius and satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$ with $G(0) = 91.$ Students were asked to find an expression for $G(t)$ and to find the internal temperature of the potato at time $t = 3$ based on this model. Students needed to employ the method of separation of variables, using the initial condition $G(0) = 91$ to resolve the constant of integration, and arrive at the particular solution $G(t) = 27 + \left(\frac{12 - t}{3}\right)^3$. Students should then have reported that the model gives an internal temperature of $G(3) = 54$ degrees Celsius for the potato at time $t = 3.$ [LO 3.5A/EK 3.5A2] This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, building notational fluency, and communicating.

Sample: 4A
Score: 9

The response earned all 9 points: 3 points in part (a), 1 point in part (b), and 5 points in part (c). In part (a) the student earned the first point for the slope with $-\frac{1}{4}(91 - 27).$ The second point was earned for the tangent line $A(t) = -16(t - 0) + 91.$ Note that the student names the tangent line $A(t)$ and is not penalized. The third point was earned for the approximation 43. Either of the numerical expressions $-16(3) + 91$ or $-48 + 91$ would have earned the third point. The student chooses to simplify and does so correctly. In part (b) the student has the correct answer of "underestimate" and supports the answer with correct reasoning. The student has the correct second derivative in line 2 on the left and states that the second derivative is always positive in line 2 on the right. The

Question 4 (continued)

student earned the point. In part (c) the student earned the first point for separation in line 2. The antiderivatives are correct in line 4 and earned the second point. The third point was earned for the constant of integration and use of the initial condition in lines 8 and 9. The fourth point was earned in line 12 for an equation involving G and t and a correct numerical value for C. As an aside, the fourth point could have been earned with an implicit equation such as $(G-27)^{1/3} = \frac{12-t}{3}$. The fifth point was earned for $G(t) = \left(-\frac{t}{3}+4\right)^3 + 27$ in line 12 along with 54 in line 4 on the right. Any of the three numerical expressions in lines 1, 2, and 3 on the right together with $G(t) = \left(-\frac{t}{3}+4\right)^3 + 27$ in line 12 would have earned the fifth point. The student chooses to simplify and does so correctly.

Sample: 4B
Score: 6

The response earned 6 points: 3 points in part (a), no point in part (b), and 3 points in part (c). In part (a) the student earned the first point for the slope with $-\frac{1}{4}(91-27)$. The second point was earned for the tangent line $y - 91 = -16t$. The third point was earned for the approximation 43. Either of the numerical expressions $-16(3) + 91$ or $-48 + 91$ would have earned the third point. The student chooses to simplify and does so correctly. In part (b) the student's answer of "underestimate" is correct, but the reason is based on an incorrect second derivative in line 3 on the left. The student did not earn the point. In part (c) the student earned the first point for separation in line 1. In line 2 the student drops a negative sign, so the second point for correct antiderivatives was not earned. Line 2 should have been: $-3(G-27)^{1/3} = t + C$. The student's equation involving antiderivatives is eligible for the remaining 3 points. In line 3 the third point was earned for the constant of integration and use of the initial condition. In line 7 the fourth point was earned for an equation involving G and t together with the consistent numerical value for C. The fifth point was not earned because, although the answer is consistent with the work, the student's value of $G(3)$ is out of the context of the problem because $152 > 91$.

Sample: 4C
Score: 3

The response earned 3 points: 3 points in part (a), no point in part (b), and no points in part (c). In part (a) the student earned the first point for the slope with $-\frac{1}{4}(91-27)$ in line 1 on the left. The second point was earned for the tangent line $H - 91 = -16(t)$. Note that the student uses H in place of y in the tangent line and is not penalized. The third point was earned for the approximation 43. Either of the numerical expressions $-16(3) + 91$ or $-48 + 91$ would have earned the third point. In part (b) the student's answer of "underestimate" is correct, but the reason is based on an incorrect second derivative of $-\frac{1}{4}$. The student did not earn the point. In part (c) there is no separation of variables, and thus, no points were earned.