2017

AP CollegeBoard

AP Calculus AB

Sample Student Responses and Scoring Commentary

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Question 5

(a) $x_P'(t) = \dfrac{2t-2}{t^2-2t+10} = \dfrac{2(t-1)}{t^2-2t+10}$

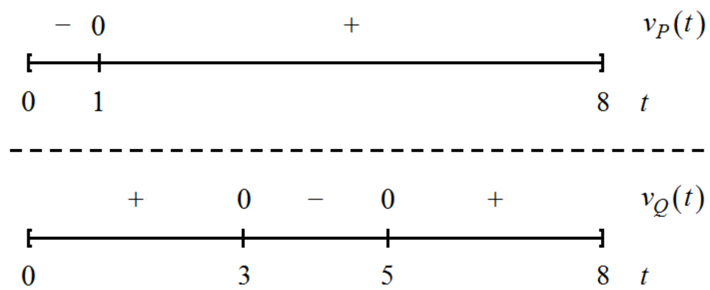
$t^2 - 2t + 10 > 0$ for all t.

$x_P'(t) = 0 \quad \Rightarrow t = 1$
$x_P'(t) < 0$ for $0 \le t < 1$.

Therefore, the particle is moving to the left for $0 \le t < 1$.

$2 : \begin{cases} 1 : x_P'(t) \\ 1 : \text{interval} \end{cases}$

(b) $v_Q(t) = (t-5)(t-3)$
$v_Q(t) = 0 \Rightarrow t = 3, t = 5$



Both particles move in the same direction for $1 < t < 3$ and $5 < t \le 8$ since $v_P(t) = x_P'(t)$ and $v_Q(t)$ have the same sign on these intervals.

$2 : \begin{cases} 1 : \text{intervals} \\ 1 : \text{analysis using } v_P(t) \text{ and } v_Q(t) \end{cases}$

Note: $1/2$ if only one interval with analysis

Note: $0/2$ if no analysis

(c) $a_Q(t) = v_Q'(t) = 2t - 8$
$a_Q(2) = 2 \cdot 2 - 8 = -4$

$a_Q(2) < 0$ and $v_Q(2) = 3 > 0$

At time $t = 2$, the speed of the particle is decreasing because velocity and acceleration have opposite signs.

$2 : \begin{cases} 1 : a_Q(2) \\ 1 : \text{speed decreasing with reason} \end{cases}$

(d) Particle Q first changes direction at time $t = 3$.

$$x_Q(3) = x_Q(0) + \int_0^3 v_Q(t)\,dt = 5 + \int_0^3 \left(t^2 - 8t + 15\right) dt$$
$$= 5 + \left[\frac{1}{3}t^3 - 4t^2 + 15t\right]_{t=0}^{t=3} = 5 + (9 - 36 + 45) = 23$$

$3 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$

NO CALCULATOR ALLOWED

5. Two particles move along the x-axis. For $0 \le t \le 8$, the position of particle P at time t is given by $x_P(t) = \ln\left(t^2 - 2t + 10\right)$, while the velocity of particle Q at time t is given by $v_Q(t) = t^2 - 8t + 15$. Particle Q is at position $x = 5$ at time $t = 0$.

(a) For $0 \le t \le 8$, when is particle P moving to the left?

$v_p(t)$ is neg.

$v_p(t) = x_p'(t) = \frac{1}{t^2-2t+10}\cdot(2t-2)$

$0 = \frac{2t-2}{t^2-2t+10}$

$2t-2=0$

$t=1$

	$t<1$	$t>1$
$2t-2$	$-$	$+$
$t^2-2t+10$	$+$	$+$
$v_p(t)$	$-$	$+$

Particle P is moving to the left on $0 \le t < 1$.

(b) For $0 \le t \le 8$, find all times t during which the two particles travel in the same direction.

$t^2-8t+15=0$

$(t-3)(t-5)=0$

$t=3,5$

	$t<1$	$1<t<3$	$3<t<5$	$t>5$
$v_p(t)$	$-$	$+$	$+$	$+$
$v_Q(t)$	$+$	$+$	$-$	$+$

	$t<3$	$3<t<5$	$t>5$
$t-3$	$-$	$+$	$+$
$t-5$	$-$	$-$	$+$
$v_Q(t)$	$+$	$-$	$+$

Particles P and Q are moving in the same direction when $1<t<3$ and $5<t\le 8$.

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NO CALCULATOR ALLOWED

(c) Find the acceleration of particle Q at time $t = 2$. Is the speed of particle Q increasing, decreasing, or neither at time $t = 2$? Explain your reasoning.

$a_Q(t) = 2t - 8$

$a_Q(2) = 2(2) - 8 = -4$

The speed of particle Q is decreasing because the particle has a positive velocity but negative acceleration when $t = 2$.

(d) Find the position of particle Q the first time it changes direction.

@ $t = 3$, particle Q first changes direction

$x_Q(t) = \int (t^2 - 8t + 15)\,dt = \frac{t^3}{3} - 4t^2 + 15t + C$

$x_Q(0) = C = 5 \Rightarrow x_Q(t) = \frac{t^3}{3} - 4t^2 + 15t + 5$

$x_Q(3) = \frac{3^3}{3} - 4(3)^2 + 15(3) + 5 = 9 - 36 + 45 + 5 = \boxed{23}$

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NO CALCULATOR ALLOWED

5. Two particles move along the x-axis. For $0 \le t \le 8$, the position of particle P at time t is given by $x_P(t) = \ln\left(t^2 - 2t + 10\right)$, while the velocity of particle Q at time t is given by $v_Q(t) = t^2 - 8t + 15$. Particle Q is at position $x = 5$ at time $t = 0$.

(a) For $0 \le t \le 8$, when is particle P moving to the left?

$V_P(t) = \frac{1}{t^2-2t+10}(2t-2)$

$\frac{2t-2}{t^2-2t+10} < 0$

$2t-2=0$

$2(t-1)=0$

$t=1$

$V_P(0.1) = \frac{0.2-2}{0.01-0.2+10}$ = negative

$V_P(5) = \frac{8}{25-10+10}$ = positive

In the interval $0 \le t < 1$ particle P moving to the left

(b) For $0 \le t \le 8$, find all times t during which the two particles travel in the same direction.

$VQ(t)=0$

$t=5,3$

$VQ(2) = 4-16+15$ = positive

$VQ(4) = 16-24+15$ = positive

$1 < t < 5$ they travel in the same direction

Continue question 5 on page 19.

NO CALCULATOR ALLOWED

(c) Find the acceleration of particle Q at time $t = 2$. Is the speed of particle Q increasing, decreasing, or neither at time $t = 2$? Explain your reasoning.

$A_Q(t) = 2t - 8$

$A_Q(2) = -4$

The speed of particle Q at t=2 is decreasing since the acceleration at t=2 is negative

(d) Find the position of particle Q the first time it changes direction.

$V_Q(t) = 0$

$t^2 - 8t + 15 = 0$

$(t-5)(t-3) = 0$

$t = 5, 3$

At t=3, t first changes direction

$\int_0^3 (t^2 - 8t + 15)\,dt = x_Q(3) - x_Q(0)$

$\left(\frac{t^3}{3} - 4t^2 + 15t\right)\Big|_0^3 = x_Q(3) - 5$

$9 - 36 + 45 = x_Q(3) - 5$

$18 + 5 = 23$

$x_Q(3) = 23$

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5. Two particles move along the x-axis. For $0 \le t \le 8$, the position of particle P at time t is given by $x_P(t) = \ln(t^2 - 2t + 10)$, while the velocity of particle Q at time t is given by $v_Q(t) = t^2 - 8t + 15$.

s(t)

Particle Q is at position $x = 5$ at time $t = 0$.

(a) For $0 \le t \le 8$, when is particle P moving to the left?

$v_p(t) = \frac{1}{t^2-2t+10}(2t-2)$

$v_p(t) = \frac{2t-2}{t^2-2t+10}$

$0 = \frac{2t-2}{t^2-2t+10}$

$t^2-2t+10 = 2t-2$

$t^2-4t+12=0$

$\left[\ln(t^2-2t+10)\right]_0^8$

$\ln(58) - \ln(10)$

$\ln\left(\frac{58}{10}\right)$

@ $x = \ln\left(\frac{58}{10}\right)$

$\left[\frac{2t-2}{t^2-2t+10}\right]_0^8$

$\frac{7}{29} - -\frac{1}{5}$

(b) For $0 \le t \le 8$, find all times t during which the two particles travel in the same direction.

@ $t = 5, 2,$ and -1

Continue question 5 on page 19.

NO CALCULATOR ALLOWED

(c) Find the acceleration of particle Q at time $t = 2$. Is the speed of particle Q increasing, decreasing, or neither at time $t = 2$? Explain your reasoning.

$a(t) = 2t - 8$

$a(2) = 2(2) - 8$

$a(2) = -4$

the speed of the particle is decreasing because the acceleration is negative and the velocity is positive @ $t=2$

$v(2) = 2^2 - 8(2) + 15$

$v(2) = 4 - 16 + 15$

$= 3$

(d) Find the position of particle Q the first time it changes direction.

$0 = t^2 - 8t + 15$

$(t-5)(t-3) = 0$

$t=5, t=3$

$\boxed{\text{at } t=5}$

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Question 5

Overview

In this problem, two particles, P and Q, are moving along the x-axis. For $0 \le t \le 8$, the position of particle P is given by $x_P(t) = \ln\left(t^2 - 2t + 10\right)$, while particle Q has position 5 at $t = 0$ and velocity $v_Q(t) = t^2 - 8t + 15$. In part (a) students were asked for those times t, $0 \le t \le 8$, when particle P's motion is to the left. Using the chain rule, students needed to find an expression for the velocity of particle P at time t by differentiating the position $x_P(t)$. By analyzing the sign of this derivative to determine those times t with $x_P'(t) < 0$, students should have concluded that particle P is moving left for $0 \le t < 1$. [LO 2.1C/EK 2.1C2-2.1C4, LO 2.3C/EK 2.3C1] In part (b) students were asked for all times t, $0 \le t \le 8$, during which both particles travel in the same direction. Using the velocity of particle P, $v_P(t) = x_P'(t)$, found in part (a), and the given velocity $v_Q(t)$ for particle Q, students needed to find those subintervals of $0 \le t \le 8$ on which both $v_P(t)$ and $v_Q(t)$ have the same sign. Students should have responded that for $1 < t < 3$ and for $5 < t \le 8$, noting that both $v_P(t)$ and $v_Q(t)$ are positive on these intervals, both particles travel in the same direction (to the right). There is no time when both velocities are negative. [LO 2.3C/EK 2.3C1] In part (c) students were asked for the acceleration of particle Q at time $t = 2$, and to determine, with explanation, whether particle Q's speed is increasing, decreasing, or neither at time $t = 2$. Students needed to differentiate $v_Q(t)$ to find that the acceleration of particle Q is given by $a_Q(t) = v_Q'(t) = 2t - 8$, and report that particle Q's acceleration at time $t = 2$ is $a_Q(2) = -4$. Students should have explained that particle Q's speed is decreasing at time $t = 2$ because the velocity and acceleration of particle Q have opposite signs at that time. [LO 2.1C/EK 2.1C2, LO 2.3C/EK 2.3C1] In part (d) students were asked to find the position of particle Q the first time it changes direction. Using the analysis of the sign of $v_Q(t)$ done in part (b), students should have concluded that the first change of direction of particle Q's motion occurs at time $t = 3$. The net change in position of particle Q across the time interval $[0, 3]$ is given by $\int_0^3 v_Q(t)\,dt$. Students needed to evaluate this integral using the Fundamental Theorem of Calculus and use the initial position of particle Q to find that particle Q's position at time $t = 3$ is $5 + \int_0^3 v_Q(t)\,dt = 23$. [LO 3.3B(b)/EK 3.3B2, LO 3.4C/EK 3.4C1] This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, building notational fluency, and communicating.

Sample: 5A
Score: 9

The response earned all 9 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and 3 points in part (d). In part (a) the student earned the first point with a correct derivative expression $x_P'(t) = \frac{1}{t^2 - 2t + 10} \cdot (2t - 2)$. The student earned the second point in the last sentence when the student writes "Particle P is moving to the left on $0 \le t < 1$." In part (b) the student earned the first point by identifying that "Particles P and Q are moving in the same direction when $1 < t < 3$ and $5 < t \le 8$" with some analysis. The student earned the second point by identifying all of the following connections between the sign of the velocity and its associated interval: $v_P(t) > 0$ for $(1, 8]$, $v_P(t) < 0$ for $[0, 1)$, $v_Q(t) < 0$ for $(3, 5)$, and $v_Q(t) > 0$ for $[0, 3)$ and $(5, 8]$. This information is included on labeled sign charts. In part (c) $a_Q(2) = 2(2) - 8$ in the second

Question 5 (continued)

line would earn the first point without simplification. The student chooses to simplify, does so correctly, and earned the first point. The student earned the second point with a correct conclusion that speed is decreasing and the reasoning using the sign of the acceleration and the sign of the velocity at $t = 2.$ In part (d) the student earned the antiderivative point in the second line with the expression $\frac{t^3}{3} - 4t^2 + 15t + C.$ The student earned the initial condition point with $x_Q(t) = \frac{t^3}{3} - 4t^2 + 15t + 5$ in the third line. $\frac{3^3}{3} - 4(3)^2 + 15(3) + 5$ or $9 - 36 + 45 + 5$ would have earned the answer point without simplification. The student chooses to simplify and does so correctly, so the student earned the answer point.

Sample: 5B
Score: 6

The response earned 6 points: 2 points in part (a), no points in part (b), 1 point in part (c), and 3 points in part (d). In part (a) the student earned the first point in the first line with a correct derivative expression $v_P(t) = \frac{1}{t^2 - 2t + 10}(2t - 2).$ The student earned the second point by "In the interval $0 \le t < 1$ particle P moving to the left." In part (b) the student's interval is incorrect, so the first point was not earned. Without correct intervals, the student is not eligible for the second point. In part (c) the student earned the first point with $A_Q(2) = -4$ in the second line. Although the student wrote that the speed of particle Q is decreasing, the student's reason is incorrect because the student does not use both the sign of the acceleration and the sign of the velocity of particle Q at $t = 2.$ The student did not earn the second point. In part (d) the student earned the initial condition point with the equation $\int_0^3 (t^2 - 8t + 15)\, dt = x_Q(3) - x_Q(0).$ The student earned the antiderivative point in the next line with the expression $\left(\frac{t^3}{3} - 4t^2 + 15t\right).$ The numeric expression $18 + 5$ would have earned the answer point without simplification. The student chooses to simplify and does so correctly, so the student earned the answer point.

Sample: 5C
Score: 3

The response earned 3 points: 1 point in part (a), no points in part (b), 2 points in part (c), and no points in part (d). In part (a) the student earned the first point in the first line with a correct derivative expression $v_P(t) = \frac{1}{t^2 - 2t + 10}(2t - 2).$ The student identifies no interval on which particle P is moving left. The student did not earn the second point. In part (b) the student did not earn the first point because the student does not include any intervals. Without correct intervals, the student is not eligible for the second point. In part (c) $a(2) = 2(2) - 8$ would have earned the first point without simplification. The student chooses to simplify and does so correctly, so the student earned the first point. The student earned the second point by identifying that the speed of particle Q is decreasing and reasoning using the sign of the acceleration and the sign of the velocity at $t = 2.$ In part (d) the student does not find the antiderivative, use the initial condition, or include an answer. Thus, the student earned no points.