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AP® CollegeBoard

AP Calculus AB

Sample Student Responses and Scoring Commentary

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Question 4

(a) $H'(6) \approx \dfrac{H(7) - H(5)}{7 - 5} = \dfrac{11 - 6}{2} = \dfrac{5}{2}$

$H'(6)$ is the rate at which the height of the tree is changing, in meters per year, at time $t = 6$ years.

$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{interpretation with units} \end{cases}$

(b) $\dfrac{H(5) - H(3)}{5 - 3} = \dfrac{6 - 2}{2} = 2$

Because H is differentiable on $3 \le t \le 5$, H is continuous on $3 \le t \le 5$.

By the Mean Value Theorem, there exists a value c, $3 < c < 5$, such that $H'(c) = 2$.

$2 : \begin{cases} 1 : \dfrac{H(5) - H(3)}{5 - 3} \\ 1 : \text{conclusion using} \\ \quad \text{Mean Value Theorem} \end{cases}$

(c) The average height of the tree over the time interval $2 \le t \le 10$ is given by $\dfrac{1}{10 - 2} \int_2^{10} H(t)\, dt$.

$$\frac{1}{8} \int_2^{10} H(t)\, dt \approx \frac{1}{8}\left(\frac{1.5 + 2}{2} \cdot 1 + \frac{2 + 6}{2} \cdot 2 + \frac{6 + 11}{2} \cdot 2 + \frac{11 + 15}{2} \cdot 3 \right)$$
$$= \frac{1}{8}(65.75) = \frac{263}{32}$$

The average height of the tree over the time interval $2 \le t \le 10$ is $\dfrac{263}{32}$ meters.

$2 : \begin{cases} 1 : \text{trapezoidal sum} \\ 1 : \text{approximation} \end{cases}$

(d) $G(x) = 50 \Rightarrow x = 1$

$$\frac{d}{dt}(G(x)) = \frac{d}{dx}(G(x)) \cdot \frac{dx}{dt} = \frac{(1 + x)100 - 100x \cdot 1}{(1 + x)^2} \cdot \frac{dx}{dt} = \frac{100}{(1 + x)^2} \cdot \frac{dx}{dt}$$

$$\left. \frac{d}{dt}(G(x)) \right|_{x=1} = \frac{100}{(1 + 1)^2} \cdot 0.03 = \frac{3}{4}$$

According to the model, the rate of change of the height of the tree with respect to time when the tree is 50 meters tall is $\dfrac{3}{4}$ meter per year.

$3 : \begin{cases} 2 : \dfrac{d}{dt}(G(x)) \\ 1 : \text{answer} \end{cases}$

Note: max $1/3$ [1-0] if no chain rule

NO CALCULATOR ALLOWED

4A
1 of 2

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

4. The height of a tree at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.

(a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

$$H'(6) \approx \frac{\Delta H(t)}{\Delta t} \approx \frac{H(7)-H(5)\,m}{(7-5)_{yr}} = \frac{(11-6)m}{(7-5)_y} = \boxed{\frac{5 \text{ meters}}{2 \text{ years}}}$$

When $t = 6$ years, the rate at which the tree is growing is $H'(6)$ meters per year

(b) Explain why there must be at least one time t, for $2 < t < 10$, such that $H'(t) = 2$.

By the MVT, as $H(t)$ is continuous and differentiable on $t \in (2,10)$, there must be $H'(c) = 2$ where $2 < c < 10$ if there exists $\frac{H(b)-H(a)}{b-a} = 2$ on the interval $(2,10)$. $\frac{H(5)-H(3)}{(5-3)\text{years}} = \frac{6m-2m}{2\text{ years}} = 2\,m/yr$

So c exists on interval $c \in (2,10)$

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(c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \le t \le 10$.

Total height: $\frac{1}{2}\left(1\,(1.5+2) + 2\,(2+6) + 2\,(6+11) + 3\,(11+15)\right)$

Average height: $\frac{1}{10_y - 2_{yr}}$ total ht $= \boxed{\frac{1}{8} \times \frac{1}{2}\left(3.5 + 2(8) + 2(17) + 3(26)\right) \text{ meters}}$

(d) The height of the tree, in meters, can also be modeled by the function G, given by $G(x) = \dfrac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?

$G(x) = \frac{100x}{1+x}$ $G(x) = 50$ $\frac{dx}{dt} = 0.03$ m/y

$x = 1$ m

$G'(x) = \dfrac{100\frac{dx}{dt}(1+x) - \left(\frac{dx}{dt}\right)100x}{(1+x)^2}$

$\dfrac{100 \times 0.03(2) - 0.03 \times 100}{4}$

$50 = \frac{100x}{1+x} \Rightarrow \cancel{50}(1+x) = \cancel{100}^{\,2}x$

$\Rightarrow 1 = x$ $x = 1$ m

$\frac{6-3}{4} = \boxed{\frac{3}{4} \text{ m/year}}$

NO CALCULATOR ALLOWED

4B
1 of 2

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

4. The height of a tree at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.

(a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

$$H'(6) = \frac{H(7) - H(5)}{7-5} = \frac{11-6}{2} = \boxed{\frac{5 \text{ meters}}{2} / \text{year}}$$

H'(6) is the rate that the tree is growing, in meters per year, at t = 6 years.

(b) Explain why there must be at least one time t, for $2 < t < 10$, such that $H'(t) = 2$.

H is twice-differentiable, which means it is also continuous. Therefore, the MVT guarantees that H'(t) = 2 since

$$\frac{H(10) - H(2)}{10 - 2} = 2$$

Continue question 4 on page 17.

4B
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(c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \leq t \leq 10$.

$$(1)\left(\frac{1.5+2}{2}\right)+(2)\left(\frac{2+6}{2}\right)+(2)\left(\frac{6+11}{2}\right)+(3)\left(\frac{11+15}{2}\right)$$

$$\frac{3.5}{2}+\frac{16}{2}+\frac{34}{2}+\frac{78}{2}$$

$$= \boxed{\frac{131.5}{2} \text{ meters}}$$

(d) The height of the tree, in meters, can also be modeled by the function G, given by $G(x) = \frac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?

$$\frac{dG}{dt} = \frac{(1+x)(100\frac{dx}{dt})-(100x)(\frac{dx}{dt})}{(1+x)^2}$$

$$\frac{(1+1)(100(0.03))-(100(1))(0.03)}{(1+1)^2}$$

$$\frac{(2)(3)-3}{4} = \frac{6-3}{4} = \boxed{\frac{3}{4} \text{ meters/year}}$$

$$50 = \frac{100x}{1+x}$$

$$50(1+x) = 100x$$

$$50+50x = 100x$$

$$50 = 50x$$

$$x = 1$$

4C
1 of 2

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

4. The height of a tree at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.

(a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

$$H'(6) = \frac{H(7) - H(5)}{7-5} = \frac{11-6}{7-5} \quad \frac{5}{2}$$

H'(6) is the rate in m/year in which the height of a tree increases.

(b) Explain why there must be at least one time t, for $2 < t < 10$, such that $H'(t) = 2$.

There must be one time t for $2 < t < 10$ that $H'(t) = 2$ b/c H(t) is continuous and differentiable

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(c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \le t \le 10$.

$$\tfrac{1}{2}(1.5+2) + \tfrac{1}{2}(2+6)2 + \tfrac{1}{2}(6+11)2 + \tfrac{1}{2}(11+15)3$$

$$= \frac{3.5}{2} + 8 + 17 + 39$$

$$= \frac{7}{2} + \frac{16}{2} + \frac{34}{2} + \frac{78}{2}$$

$$= \frac{135}{2}$$

(d) The height of the tree, in meters, can also be modeled by the function G, given by $G(x) = \dfrac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?

$$r = \frac{d}{2}$$

$$G'(x) = \frac{(1+x)(100) - 100x}{(1+x)^2}$$

$$A = bh + \pi r^2$$

$$A = dh + \pi\left(\frac{d}{2}\right)^2$$

$$A = x\left(\frac{100x}{1+x}\right) + \pi\left(\frac{x}{2}\right)^2$$

Question 4

Overview

The context of this problem is a tree, the height of which at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are provided in a table. In part (a) students were asked to use the tabular data to estimate $H'(6)$ and then to interpret the meaning of $H'(6)$, using correct units, in the context of the problem. The correct response should estimate the derivative value using a difference quotient, drawing from data in the table that most tightly bounds $t = 6$. In part (b) students were asked to explain why there must be at least one time t, for $2 < t < 10$, such that $H'(t) = 2$. A correct response should demonstrate that the Mean Value Theorem applies to H on the interval $[3, 5]$, over which the average rate of change of H (using data from the table) is $\frac{6-2}{2} = 2$. In part (c) students were asked to use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \le t \le 10$. A correct response should demonstrate that the average height of the tree for $2 \le t \le 10$ is given by dividing the definite integral of H across the interval by the width of the interval. The value of the integral $\int_2^{10} H(t)\,dt$ is to be approximated using a trapezoidal sum and data in the table. In part (d) students were given another model for the tree's height, in meters, $G(x) = \frac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. It is further given that when the tree is 50 meters tall, it is growing so that the diameter at the base of the tree is increasing at the rate of 0.03 meter per year. Using this model, students were asked to find the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall. A correct response should apply the chain rule to obtain that $\frac{dG}{dt} = \frac{dG}{dx} \cdot \frac{dx}{dt}$. The derivative expression $\frac{dG}{dx}$ can be obtained from the given expression for $G(x)$ using derivative rules (e.g., the quotient rule) and the value of $\frac{dx}{dt}$ at the instant in question provided in the problem statement.

For part (a) see LO 2.1B/EK 2.1B1, LO 2.3A/EK 2.3A1, LO 2.3A/EK 2.3A2. For part (b) see LO 2.4A/EK 2.4A1. For part (c) see LO 3.2B/EK 3.2B2, LO 3.4B/EK 3.4B1. For part (d) see LO 2.1C/EK 2.1C3, LO 2.1C/EK 2.1C4, LO 2.3C/EK 2.3C2. This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, connecting multiple representations, building notational fluency, and communicating.

Sample: 4A
Score: 9

The response earned all 9 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and 3 points in part (d). In part (a) the response would have earned the first point for $\frac{(11-6)}{(7-5)}$ with no simplification. In this case, correct simplification to $\frac{5}{2}$ earned the point. Although not required for the first point, the answer includes the correct units of "meters/years" which is considered for the second point. The response earned the second point for the interpretation that includes the three necessary elements: an interpretation of H' as a rate in the context of the tree, correct units of meters per year, and an interpretation of the input "6" as the moment in time of $t = 6$ years. In part (b) the response earned the first point for the difference quotient $\frac{H(5) - H(3)}{(5-3)}$ that appears in line 4. The first point does not require the substitution of function values and simplification that follows; this work is considered in the context of the second point. The response earned the second point for the explanation using the Mean Value Theorem (MVT) and the difference quotient on the interval $[3, 5]$. The response explicitly states in line 1 that $H(t)$ is continuous which is necessary to earn the point. The units displayed are correct but not required to earn either the first or second points. In part (c) the response earned the first point for the trapezoidal sum labeled "Total height" with no simplification. The response earned the second point with the boxed answer with no simplification. Note that both points were earned in part (c) without simplification of numerical answers. The units given with the average height are correct but not required to earn the second point. In part (d) the response earned the first 2 points with the correct derivative in line 2: $\dfrac{100\frac{dx}{dt}(1+x)-\left(\frac{dx}{dt}\right)100x}{(1+x)^2}$. The use of $G'(x)$ rather than $\frac{dG}{dt}$ notation does not impact the points earned. The response would have earned the third point for $\dfrac{100\times 0.03(2)-0.03\times 100}{4}$ with no simplification. In this case, correct simplification to $\frac{3}{4}$ earned the third point. The response includes correct units that are not required to earn the third point.

Sample: 4B
Score: 6

The response earned 6 points: 2 points in part (a), no points in part (b), 1 point in part (c), and 3 points in part (d). In part (a) the response would have earned the first point for $\frac{11-6}{2}$ with no simplification. In this case, correct simplification to $\frac{5}{2}$ earned the point. Although not required for the first point, the answer includes the correct units of meters/year which is considered for the second point. The response earned the second point for the interpretation that includes the three necessary elements: an interpretation of H' as a rate in the context of the tree, correct units of meters per year, and an interpretation of the input "6" as the moment in time of $t = 6$ years. In part (b) the response did not earn the first point because the difference quotient does not use the interval $[3, 5]$ that results in a secant slope of 2. The response did not earn the second point because, although the Mean Value Theorem (MVT) is cited along with the continuity of H, there is no explanation connecting the Mean Value Theorem to the values of $H(t)$ in the table. In part (c) the response earned the first point for the trapezoidal sum in line 1 with no simplification. The arithmetic and simplification that follow are considered for the second point. The second point was not earned because the sum is not multiplied by $\frac{1}{8}$ to find the average height of the tree on the interval $[2, 10]$. In part (d) the response earned the first 2 points with the correct derivative in line 1 on the left: $\dfrac{dG}{dt}=\dfrac{(1+x)\left(100\frac{dx}{dt}\right)-(100x)\left(\frac{dx}{dt}\right)}{(1+x)^2}$. The response would have earned the third point for $\dfrac{(1+1)(100(0.03))-(100(1))(0.03)}{(1+1)^2}$ in line 2 on the left with no simplification. In this case, correct simplification to $\frac{3}{4}$ earned the third point. The response includes correct units that are not required to earn the third point.

Question 4 (continued)

Sample: 4C
Score: 3

The response earned 3 points: 1 point in part (a), no points in part (b), 1 point in part (c), and 1 point in part (d). In part (a) the response would have earned the first point for $\frac{11-6}{7-5}$ with no simplification. In this case, correct simplification to $\frac{5}{2}$ earned the point. The response did not earn the second point because the interpretation of $H'(6)$ does not include an interpretation of the input "6" as the moment in time of $t = 6$ years. In part (b) the response did not earn the first point because no difference quotient is given. The response is not eligible for the second point because the explanation given does not reference the interval $[3, 5]$ that results in the secant slope of 2. Although the hypotheses of the Mean Value Theorem are stated, the conclusion is not. In part (c) the response earned the first point for the trapezoidal sum in line 1 with no simplification. The arithmetic and simplification that follow are considered for the second point. The response did not earn the second point. There is an arithmetic error in line 3, and the sum is not multiplied by $\frac{1}{8}$ to find the average height of the tree on the interval $[2, 10]$. In part (d) the response earned 1 of the first 2 points for the correct derivative of G with respect to x in line 2 on the right. Because the derivative does not include the chain rule, the response is not eligible for additional points in part (d).