2019

AP® CollegeBoard

AP® Calculus AB

Sample Student Responses and Scoring Commentary

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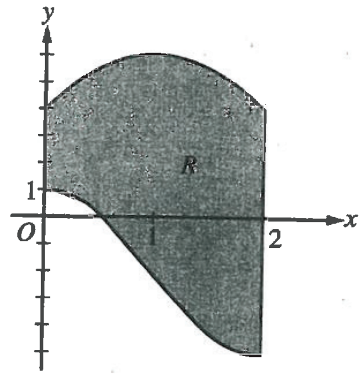
Question 5

(a) $\int_0^2 (h(x) - g(x))\,dx = \int_0^2 \left(\left(6 - 2(x-1)^2\right) - \left(-2 + 3\cos\left(\frac{\pi}{2}x\right)\right)\right) dx$ $= \left[\left(6x - \frac{2}{3}(x-1)^3\right) - \left(-2x + \frac{6}{\pi}\sin\left(\frac{\pi}{2}x\right)\right)\right]_{x=0}^{x=2}$ $= \left(\left(12 - \frac{2}{3}\right) - (-4 + 0)\right) - \left(\left(0 + \frac{2}{3}\right) - (0 + 0)\right)$ $= 12 - \frac{2}{3} + 4 - \frac{2}{3} = \frac{44}{3}$ The area of R is $\frac{44}{3}$.	$4: \begin{cases} 1: \text{integrand} \\ 1: \text{antiderivative of } 3\cos\left(\frac{\pi}{2}x\right) \\ 1: \text{antiderivative of remaining terms} \\ 1: \text{answer} \end{cases}$
(b) $\int_0^2 A(x)\,dx = \int_0^2 \frac{1}{x+3}\,dx$ $= \left[\ln(x+3)\right]_{x=0}^{x=2} = \ln 5 - \ln 3$ The volume of the solid is $\ln 5 - \ln 3$.	$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$
(c) $\pi\int_0^2 \left((6 - g(x))^2 - (6 - h(x))^2\right) dx$	$3: \begin{cases} 1: \text{limits and constant} \\ 1: \text{form of integrand} \\ 1: \text{integrand} \end{cases}$



5. Let R be the region enclosed by the graphs of $g(x) = -2 + 3\cos\left(\frac{\pi}{2}x\right)$ and $h(x) = 6 - 2(x-1)^2$, the y-axis, and the vertical line $x = 2$, as shown in the figure above.

(a) Find the area of R.

$$\text{Area} = \int_0^2 h(x) - g(x)\,dx$$

$$= \int_0^2 6 - 2(x-1)^2\,dx - \int_0^2 -2 + 3\cos\left(\tfrac{\pi}{2}x\right)dx$$

$$= 6x - \tfrac{2}{3}(x-1)^3 \Big|_0^2 - \left(-2x + \tfrac{6}{\pi}\sin\left(\tfrac{\pi}{2}x\right)\right)\Big|_0^2$$

$$= \left(\left[12 - \tfrac{2}{3}(1)\right] - \left[0 + \tfrac{2}{3}\right]\right) - \left(\left[-4 + \tfrac{6}{\pi}(0)\right] - [0 + 0]\right)$$

$$= 12 - \tfrac{4}{3} + 4 = 16 - \tfrac{4}{3}$$

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(b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis has area $A(x) = \frac{1}{x+3}$. Find the volume of the solid.

$$V = \int_0^2 A(x)\,dx = \int_0^2 \frac{1}{x+3}\,dx = \ln|x+3|\Big|_0^2 = \ln|5| - \ln|3|$$
$$= \ln 5 - \ln 3$$

(c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 6$.

$$V = \int_0^2 \pi[(6-g(x))^2 - (6-h(x))^2]\,dx$$

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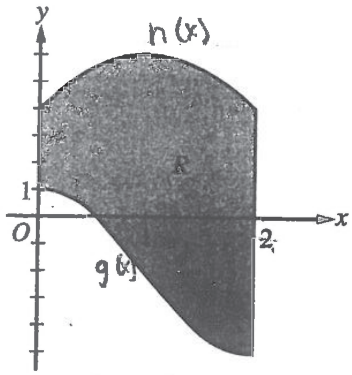

5. Let R be the region enclosed by the graphs of $g(x) = -2 + 3\cos\left(\frac{\pi}{2}x\right)$ and $h(x) = 6 - 2(x-1)^2$, the y-axis, and the vertical line $x = 2$, as shown in the figure above.

(a) Find the area of R.

$$\int_0^2 (h(x) - g(x))\,dx$$

$$\int_0^2 \left(\left(6 - 2(x-1)^2\right) - \left(-2 + 3\cos\left(\tfrac{\pi}{2}x\right)\right)\right)dx$$

$$6x - \tfrac{2}{3}(x-1)^3 \Big]_0^2 - \left(-2x + \tfrac{3\pi}{2}\sin\left(\tfrac{\pi}{2}x\right)\right)\Big]_0^2$$

$$12 - (-4) = 12 + 4 = 16$$

Continue question 5 on page 19.

NO CALCULATOR ALLOWED

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(b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis has area $A(x) = \frac{1}{x+3}$. Find the volume of the solid.

$$V = \pi \int_0^2 (A(x))dx = \pi \int_0^2 \frac{1}{x+3} dx$$

$$V = \pi \cdot \ln|x+3| \Big]_0^2 = \pi(\ln|5| - \ln|3|)$$

(c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 6$.

$$\pi \int_0^2 (6 - g(x))^2 - (6 - h(x))^2 \, dx$$

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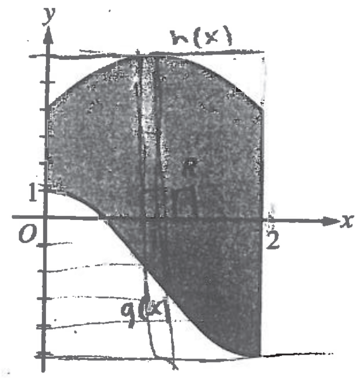

5. Let R be the region enclosed by the graphs of $g(x) = -2 + 3\cos\left(\frac{\pi}{2}x\right)$ and $h(x) = 6 - 2(x-1)^2$, the y-axis, and the vertical line $x = 2$, as shown in the figure above.

(a) Find the area of R.

$$A = \int_0^2 (h(x) - g(x))^2 \, dx$$

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NO CALCULATOR ALLOWED

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(b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis has area $A(x) = \frac{1}{x+3}$. Find the volume of the solid.

$$V = \int_0^2 \frac{1}{x+3} (h(x) - g(x))^2 \, dx$$

(c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 6$.

$$V = \pi \int_0^2 (g(x) - 6)^2 - (h(x) - 6)^2 \, dx$$

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Question 5

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem R is identified as the region enclosed by the graphs of $g(x) = -2 + 3\cos\left(\frac{\pi}{2}x\right)$ and $h(x) = 6 - 2(x-1)^2$, the y-axis, and the vertical line $x = 2$.

In part (a) students were asked to find the area of R. A response should demonstrate the area interpretation of definite integrals and compute the area of R as $\int_0^2 (h(x) - g(x))\,dx$. Students should find an antiderivative for $h(x) - g(x)$ and apply the Fundamental Theorem of Calculus to evaluate the integral.

In part (b) students were asked to find the volume of a solid having R as its base and for which at each x, the cross section perpendicular to the x-axis has area $A(x) = \frac{1}{x+3}$. A response should demonstrate that the volume is found by integrating the cross-sectional area function across the interval $0 \le x \le 2$. As before, the Fundamental Theorem of Calculus should be employed to evaluate $\int_0^2 A(x)\,dx$.

In part (c) students were asked to write an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 6$. A response should demonstrate that the volume is found by integrating the cross-sectional area function across the interval $0 \le x \le 2$. In this case, however, the cross section at x is a "washer" with outer radius $6 - g(x)$ and inner radius $6 - h(x)$, so the area of the cross section at x can be expressed using the familiar formula for the area of a circle.

For part (a) see LO CHA-5.A/EK CHA-5.A.1, LO FUN-6.C/EK FUN-6.C.2, LO FUN-6.B/EK FUN-6.B.3. For part (b) see LO CHA-5.B/EK CHA-5.B.3, LO FUN-6.B/EK FUN-6.B.3. For part (c) see LO CHA-5.C/EK CHA-5.C.4. This problem incorporates the following Mathematical Practices: Practice 1: Implementing Mathematical Processes and Practice 4: Communication and Notation.

Sample: 5A
Score: 9

The response earned 9 points: 4 points in part (a), 2 points in part (b), and 3 points in part (c). In part (a) the response earned the first point in line 1 with $\int_0^2 h(x) - g(x)\,dx$. The missing parentheses in the integrand do not impact earning the point. The antiderivative of the cosine term, $-\left(+\frac{6}{\pi}\sin\left(\frac{\pi}{2}x\right)\right)$, in line 3 is correct, and the response earned the second point. The antiderivative of the remaining terms, $6x - \frac{2}{3}(x-1)^3 - (-2x)$, in line 3 is correct, and the response earned the third point. The response earned the fourth point in the last line with $16 - \frac{4}{3}$. Note that the response could have ended at line 4 because numerical simplification is not required for the fourth point. Although this response does so, evaluation of trigonometric functions is also not required for the fourth

Question 5 (continued)

point. In part (b) the response earned the first point with $\int_0^2 A(x)\,dx = \int_0^2 \frac{1}{x+3}\,dx$ in line 1. The first definite integral is sufficient to earn the point. The second point was earned with the answer $\ln 5 - \ln 3$ in line 2. Note that the absolute value is not needed in this case because on the interval $[0, 2]$, $x + 3 > 0$. In part (c) the response earned all 3 points with the expression $\int_0^2 \pi\left[(6 - g(x))^2 - (6 - h(x))^2\right] dx$ because the limits, constant, and integrand are correct.

Sample: 5B
Score: 6

The response earned 6 points: 2 points in part (a), 1 point in part (b), and 3 points in part (c). In part (a) the response earned the first point in line 1 with $\int_0^2 (h(x) - g(x))\,dx.$ The antiderivative of the cosine term, $-\left(+\frac{3\pi}{2}\sin\left(\frac{\pi}{2}x\right)\right)$, in line 3 is incorrect because the factor of $\frac{3\pi}{2}$ should be $\frac{6}{\pi}$. The second point was not earned. The antiderivative of the remaining terms, $6x - \frac{2}{3}(x-1)^3 - (-2x)$, in line 3 is correct, and the response earned the third point. Because both the first point and 1 of the second and third points were earned, the response is eligible for the fourth point if the answer is consistent with previous work. The response has an error on the left side of the equation in line 4 because the expression $12 - (-4)$ should be $\left(12 - \frac{4}{3}\right) - (-4)$. The fourth point was not earned. In part (b) the response earned the first point with $\int_0^2 (A(x))\,dx$ in line 1. The multiplication by π is incorrect and produces an incorrect answer. The second point was not earned. In part (c) the response earned all 3 points with the expression $\pi\int_0^2 (6 - g(x))^2 - (6 - h(x))^2\,dx$ because the limits, constant, and integrand are correct. The missing parentheses in the integrand do not impact earning the point.

Sample: 5C
Score: 3

The response earned 3 points: no points in part (a), no points in part (b), and 3 points in part (c). In part (a) the response has a definite integral $\int_0^2 (h(x) - g(x))^2\,dx$ with an incorrect integrand, so the first point was not earned. A response that includes a definite integral that when evaluated does not represent the area of R is not eligible for points in part (a). As a result, although this response presents no additional work, the response is not eligible for any points in part (a). In part (b) the response has a definite integral $\int_0^2 \frac{1}{x+3}(h(x) - g(x))^2\,dx$ with an incorrect factor of $(h(x) - g(x))^2$ in the integrand. The first point was not earned, and the response is not eligible for the second point because of the form of the integrand presented. In part (c) the response earned all 3 points with the expression $\pi\int_0^2 (g(x) - 6)^2 - (h(x) - 6)^2\,dx$ because the limits, constant, and integrand are correct. Because each of the two terms in the integrand are squared, this response presents an integrand that is equivalent to $(6 - g(x))^2 - (6 - h(x))^2$. The missing parentheses in the integrand do not impact earning the point.