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AP® CollegeBoard

AP® Calculus AB

Sample Student Responses and Scoring Commentary

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Question 6

(a) $h'(2) = \dfrac{2}{3}$

1 : answer

(b) $a'(x) = 9x^2 h(x) + 3x^3 h'(x)$

$a'(2) = 9 \cdot 2^2 h(2) + 3 \cdot 2^3 h'(2) = 36 \cdot 4 + 24 \cdot \dfrac{2}{3} = 160$

$3 : \begin{cases} 1 : \text{form of product rule} \\ 1 : a'(x) \\ 1 : a'(2) \end{cases}$

(c) Because h is differentiable, h is continuous, so $\lim\limits_{x \to 2} h(x) = h(2) = 4$.

Also, $\lim\limits_{x \to 2} h(x) = \lim\limits_{x \to 2} \dfrac{x^2 - 4}{1 - (f(x))^3}$, so $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{1 - (f(x))^3} = 4$.

Because $\lim\limits_{x \to 2} \left(x^2 - 4\right) = 0$, we must also have $\lim\limits_{x \to 2} \left(1 - (f(x))^3\right) = 0$.

Thus $\lim\limits_{x \to 2} f(x) = 1$.

Because f is differentiable, f is continuous, so $f(2) = \lim\limits_{x \to 2} f(x) = 1$.

Also, because f is twice differentiable, f' is continuous, so $\lim\limits_{x \to 2} f'(x) = f'(2)$ exists.

Using L'Hospital's Rule,

$$\lim_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3} = \lim_{x \to 2} \frac{2x}{-3(f(x))^2 f'(x)} = \frac{4}{-3(1)^2 \cdot f'(2)} = 4.$$

Thus $f'(2) = -\dfrac{1}{3}$.

$4 : \begin{cases} 1 : \lim\limits_{x \to 2} \dfrac{x^2 - 4}{1 - (f(x))^3} = 4 \\ 1 : f(2) \\ 1 : \text{L'Hospital's Rule} \\ 1 : f'(2) \end{cases}$

(d) Because g and h are differentiable, g and h are continuous, so $\lim\limits_{x \to 2} g(x) = g(2) = 4$ and $\lim\limits_{x \to 2} h(x) = h(2) = 4$.

Because $g(x) \le k(x) \le h(x)$ for $1 < x < 3$, it follows from the squeeze theorem that $\lim\limits_{x \to 2} k(x) = 4$.

Also, $4 = g(2) \le k(2) \le h(2) = 4$, so $k(2) = 4$.

Thus k is continuous at $x = 2$.

1 : continuous with justification

NO CALCULATOR ALLOWED

6A 1 of 2

6. Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x - 2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

(a) Find $h'(2)$.

$y = \frac{2}{3}(x-2) + 4$

point $(2,4)$ slope $\frac{2}{3}$

$h'(2) = 2/3$

(b) Let a be the function given by $a(x) = 3x^3 h(x)$. Write an expression for $a'(x)$. Find $a'(2)$.

$a'(x) = h'(x) \cdot 3x^3 + 9x^2 h(x)$

$a'(2) = h'(2) \cdot 3(2)^3 + 9(2)^2 h(2)$

$a'(2) = \frac{2}{3} \cdot 3 \cdot 2^3 + 9 \cdot 2^2 \cdot 4$

Continue question 6 on page 21.

NO CALCULATOR ALLOWED

6A 2 of 2

(c) The function h satisfies $h(x) = \dfrac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$. It is known that $\lim_{x\to 2} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x\to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

$$\lim_{x\to 2} \frac{x^2-4}{1-(f(x))^3} = \lim_{x\to 2} \frac{2x}{-3(f(x))^2 \cdot f'(x)}$$

$h(2)=4$

$\therefore \lim_{x\to 2} h(x)$ MUST equal 4

$\lim_{x\to 2} x^2-4=0$

$\lim_{x\to 2} 1-(f(x))^3=0$ if $f(2)=1$

$\therefore$ Indeterminate form $\frac{0}{0}$ and L'Hopital's rule applies SUCH THAT $f(2)=1$

$$= \frac{2(2)}{-3(1)^2 \cdot f'(2)}$$

$$= \frac{4}{-3 \cdot f'(2)}$$

$f'(2)$ MUST equal $-1/3$ so that $\lim_{x\to 2} h(x)=4$

$$= \frac{4}{-3 \cdot -1/3}$$

$$= \frac{4}{1}$$

$\lim_{x\to 2} h(x)=4$ such that $f(2)=1$ and $f'(2)=-1/3$

(d) It is known that $g(x) \leq h(x)$ for $1 < x < 3$. Let k be a function satisfying $g(x) \leq k(x) \leq h(x)$ for $1 < x < 3$. Is k continuous at $x = 2$? Justify your answer.

g and h given as twice-differentiable $\therefore$ g and h are continuous for $1<x<3$

$g(x) \leq k(x) \leq h(x)$

$\lim_{x\to 2} g(x) = 4$ because g continuous and $g(2)=4$

$\lim_{x\to 2} h(x) = 4$ because h continuous and $h(2)=4$

By Squeeze theorem, $\lim_{x\to 2} k(x) = 4$

$g(2)=4=h(2)$

k is between or equal to g and h for $1<x<3$, given meaning $k(2)$ must equal 4, and $\lim_{x\to 2} k(x)=4$ so $k(x)$ is continuous at $x=2$

NO CALCULATOR ALLOWED

6B 1 of 2

6. Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x - 2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

(a) Find $h'(2)$.

$y = 4 + \frac{2}{3}(x-2)$

$y - 4 = \frac{2}{3}(x-2)$

$h'(2) = \frac{2}{3}$

(b) Let a be the function given by $a(x) = 3x^3 h(x)$. Write an expression for $a'(x)$. Find $a'(2)$.

$a'(x) = h(x)(9x^2) + (3x^3)h'(x)$

$a'(2) = h(2)(9(2)^2) + (3(2)^3)h'(2)$

$a'(2) = 4(9(2)^2) + (3(2)^3)\left(\frac{2}{3}\right)$

Continue question 6 on page 21.

NO CALCULATOR ALLOWED

6B 2 of 2

(c) The function h satisfies $h(x) = \dfrac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$. It is known that $\lim\limits_{x \to 2} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim\limits_{x \to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

$$\lim_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3} = 4$$

$$\lim_{x \to 2} \frac{2x}{-3(f(x))^2 f'(x)} = 4$$

$$\frac{2(2)}{-3(f(2))^2 f'(2)} = 4$$

$$\frac{4}{-3(f(2))^2 f'(2)} = 4$$

(d) It is known that $g(x) \leq h(x)$ for $1 < x < 3$. Let k be a function satisfying $g(x) \leq k(x) \leq h(x)$ for $1 < x < 3$. Is k continuous at $x = 2$? Justify your answer.

K is continuous at x=2 given that K(x) is less than or equal too h(x) on 1< x < 3 and greater than or equal too g(x) on 1<x<3, so K(x) must simply be equal to g(x) and h(x) at x=2 since both g(x) and h(x) = 4 at x=2

GO ON TO THE NEXT PAGE.

6C 1 of 2

6. Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x - 2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

(a) Find $h'(2)$.

$h'(2) = 2/3$

(b) Let a be the function given by $a(x) = 3x^3 h(x)$. Write an expression for $a'(x)$. Find $a'(2)$.

$a'(x) = (3x^3)(h'(x)) + (9x^2)(h(x))$

$a'(2) = (3(2)^3)(h'(2)) + (9(2)^2)(h(2))$

$= 8 \cdot h'(2) + 36 \cdot 4$

$= 8 \cdot h'(2) + 144$

$= 8 \cdot \frac{2}{3} + 144$

$\frac{16}{3} + \frac{432}{3}$

$\frac{448}{3}$

Continue question 6 on page 21.

(c) The function h satisfies $h(x) = \dfrac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$. It is known that $\lim_{x \to 2} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x \to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

$$\frac{2x}{-3(f(x))^2 f'(x)}$$

(d) It is known that $g(x) \leq h(x)$ for $1 < x < 3$. Let k be a function satisfying $g(x) \leq k(x) \leq h(x)$ for $1 < x < 3$. Is k continuous at $x = 2$? Justify your answer.

k is continuos at 2 because using IVT, it states that there is a value for $1<x<3$ that equals that due to MVT. $g(2)=h(2)=4$ so yes.

Question 6

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

This problem introduces three twice-differentiable functions f, g, and h. It is given that $g(2) = h(2) = 4$, and the line $y = 4 + \frac{2}{3}(x - 2)$ is tangent at $x = 2$ to both the graph of g and the graph of h.
In part (a) students were asked to find $h'(2)$. A response should demonstrate the interpretation of the derivative as the slope of a tangent line and answer with the slope of the line $y = 4 + \frac{2}{3}(x - 2)$.

In part (b) the function a given by $a(x) = 3x^3 h(x)$ is defined, and students were asked for an expression for $a'(x)$ and the value of $a'(2)$. A response should demonstrate facility with the product rule for differentiation.

In part (c) it is given that the function h satisfies $h(x) = \dfrac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$ and that $\lim_{x\to 2} h(x)$ can be evaluated using L'Hospital's Rule. Students were then asked to find $f(2)$ and $f'(2)$. A response should observe that the differentiability of h implies that h is continuous so that $\lim_{x\to 2} h(x) = h(2) = 4$. Because $\lim_{x\to 2}\left(x^2 - 4\right) = 0$, and $\lim_{x\to 2} h(x)$ can be evaluated, as is given, it must be that $\lim_{x\to 2}\left(1 - (f(x))^3\right) = 0$, as well. Using properties of limits, students could conclude that $\lim_{x\to 2} f(x) = 1$. Finally, an application of L'Hospital's Rule to $\lim_{x\to 2} \dfrac{x^2 - 4}{1 - (f(x))^3}$, combined with the chain rule to differentiate $(f(x))^3$, yields an equation that can be solved for $f'(2)$.

In part (d) students were given that $g(x) \le h(x)$ for $1 < x < 3$ and that k is a function satisfying $g(x) \le k(x) \le h(x)$ for $1 < x < 3$. Students were asked to decide, with justification, whether k is continuous at $x = 2$. A response should observe that the differentiability of g and h implies that these functions are continuous, so the limits as x approaches 2 of each of g and h match the value $g(2) = h(2) = 4$. From the inequality $g(2) \le k(2) \le h(2)$ it follows that $k(2) = 4$, and the squeeze theorem applies to show that k is continuous at $x = 2$.

For part (a) see LO CHA-2.C/EK CHA-2.C.1. For part (b) see LO FUN-3.B/EK FUN-3.B.1. For part (c) see LO LIM-2.A/EK LIM-2.A.2, LO LIM-4.A/EK LIM-4.A.2. For part (d) see LO LIM-1.E/EK LIM.1.E.2. This problem incorporates the following Mathematical Practices: Practice 1: Implementing Mathematical Processes, Practice 3: Justification, and Practice 4: Communication and Notation.

Sample: 6A
Score: 9

The response earned 9 points: 1 point in part (a), 3 points in part (b), 4 points in part (c), and 1 point in part (d). In part (a) the response earned the point for the value of $h'(2) = \frac{2}{3}$ in line 3. In part (b) the response earned both the first and second points for the derivative $a'(x) = h'(x) \cdot 3x^3 + 9x^2 h(x)$ in line 1. The response earned the third

Question 6 (continued)

point for the numerical expression $a'(2) = \frac{2}{3} \cdot 3 \cdot 2^3 + 9 \cdot 2^2 \cdot 4$ in line 3. Numerical simplification is not required. In part (c) the response earned the first point for the verbal connection "$\lim_{x\to 2} h(x)$ MUST equal 4" in lines 2 and 3 on the right. The response would have earned the second point for the equation $\lim_{x\to 2} 1 - (f(x))^3 = 0$ in line 3 on the left and presenting $f(2) = 1$ in line 3 on the left. The response correctly restates $f(2) = 1$ in line 6 on the left and earned the second point with the final restatement of $f(2) = 1$ in the circled statement in the last line. The response earned the third point for L'Hospital's Rule with the expression $\lim_{x\to 2} \frac{2x}{-3(f(x))^2 \cdot f'(x)}$ in line 1 in the middle. The response would have earned the fourth point with "$f'(2)$ MUST equal $\frac{-1}{3}$" in line 4 on the right. The response earned the fourth point with the restatement of $f'(2) = \frac{-1}{3}$ in the circled statement in the last line. In part (d) the response earned the point for concluding that $k(x)$ is continuous at $x = 2$ in line 8 with the following justification: stating g and h are continuous in line 1, evaluating $\lim_{x\to 2} g(x) = 4$ in line 4 and $\lim_{x\to 2} h(x) = 4$ in line 5, concluding $\lim_{x\to 2} k(x) = 4$ in line 6, and concluding $k(2) = 4$ because $g(2) = 4 = h(2)$ in lines 7 and 8. Although not required, the response correctly states use of the squeeze theorem.

Sample: 6B
Score: 6

The response earned 6 points: 1 point in part (a), 3 points in part (b), 2 points in part (c), and no point in part (d). In part (a) the response earned the point for the value of $h'(2) = \frac{2}{3}$ in line 1 on the right. In part (b) the response earned the first and second points for the derivative $a'(x) = h(x)\left(9x^2\right) + \left(3x^3\right)h'(x)$ in line 1. The response earned the third point for the numerical expression $a'(2) = 4\left(9(2)^2\right) + \left(3(2)^3\right)\left(\frac{2}{3}\right)$ in line 3. Numerical simplification is not required. In part (c) the response earned the first point for the equation $\lim_{x\to 2} \frac{x^2 - 4}{1 - (f(x))^3} = 4$ in line 1 on the left. Although not required for the first point, application of L'Hospital's Rule resulting in the equation $\lim_{x\to 2} \frac{2x}{-3(f(x))^2 f'(x)} = 4$ in line 1 on the right would also have earned the first point. The response did not earn the second point because no value is given for $f(2)$. The response earned the third point for the expression $\lim_{x\to 2} \frac{2x}{-3(f(x))^2 f'(x)}$ in line 1 on the right. The response did not earn the fourth point because no value is given for $f'(2)$. In part (d) the response correctly concludes that k is continuous at $x = 2$. The response did not earn the point because the justification is not sufficient. The response does not state that g and h are continuous, does not evaluate $\lim_{x\to 2} g(x) = 4$ and $\lim_{x\to 2} h(x) = 4$, and does not conclude $\lim_{x\to 2} k(x) = 4 = k(2)$.

Question 6 (continued)

Sample: 6C
Score: 3

The response earned 3 points: 1 point in part (a), 2 points in part (b), no points in part (c), and no point in part (d). In part (a) the response earned the point for the value of $h'(2) = \frac{2}{3}$. In part (b) the response earned the first and second points for the derivative $a'(x) = \left(3x^3\right)(h'(x)) + \left(9x^2\right)(h(x))$ in line 1. The response did not earn the third point because of an incorrect simplification in the numerical expression of $a'(2)$ with rewriting $\left(3(2)^3\right)$ in line 2 as 8 in line 3. In part (c) the response did not earn the first point because there is no connection between $\lim_{x \to 2} h(x)$ and $4.$ The response did not earn the second point because no value is given for $f(2).$ Although the response attempts to apply L'Hospital's Rule in line 1, the response did not earn the third point because of a lack of limit notation: The use of $\lim_{x \to 2}$ is not presented with the quotient. The response did not earn the fourth point because no value is given for $f'(2).$ In part (d) the response correctly concludes that k is continuous at $2.$ The response did not earn the point because the justification is not sufficient. The response attempts to apply both the "IVT" (Intermediate Value Theorem) and the "MVT" (Mean Value Theorem) rather than the squeeze theorem.