2021

AP CollegeBoard

AP® Calculus AB

Sample Student Responses and Scoring Commentary

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Free Response Question 3

- ☑ Scoring Guideline
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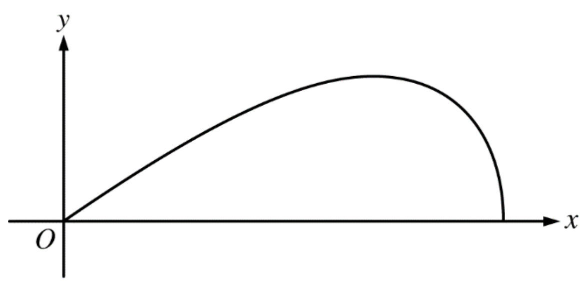
Part B (AB or BC): Graphing calculator not allowed
Question 3 — 9 points

General Scoring Notes

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.



A company designs spinning toys using the family of functions $y = cx\sqrt{4 - x^2}$, where c is a positive constant. The figure above shows the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$, for some c. Each spinning toy is in the shape of the solid generated when such a region is revolved about the x-axis. Both x and y are measured in inches.

	Model Solution	Scoring	
(a)	Find the area of the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$ for $c = 6$.		
	$6x\sqrt{4 - x^2} = 0 \Rightarrow x = 0,\ x = 2$ Area $= \int_0^2 6x\sqrt{4 - x^2}\,dx$	Integrand	**1 point**
	Let $u = 4 - x^2$. $du = -2x\,dx \Rightarrow -\frac{1}{2}\,du = x\,dx$ $x = 0 \Rightarrow u = 4 - 0^2 = 4$ $x = 2 \Rightarrow u = 4 - 2^2 = 0$ $\int_0^2 6x\sqrt{4 - x^2}\,dx = \int_4^0 6\left(-\frac{1}{2}\right)\sqrt{u}\,du = -3\int_4^0 u^{1/2}\,du = 3\int_0^4 u^{1/2}\,du$ $= 2u^{3/2}\Big\vert_{u=0}^{u=4} = 2 \cdot 8 = 16$	Antiderivative	**1 point**
	The area of the region is 16 square inches.	Answer	**1 point**

Scoring notes:

- Units are not required for any points in this question and are not read if presented (correctly or incorrectly) in any part of the response.
- The first point is earned for presenting $cx\sqrt{4 - x^2}$ or $6x\sqrt{4 - x^2}$ as the integrand in a definite integral. Limits of integration (numeric or alphanumeric) must be presented (as part of the definite integral) but do not need to be correct in order to earn the first point.
- If an indefinite integral is presented with an integrand of the correct form, the first point can be earned if the antiderivative (correct or incorrect) is eventually evaluated using the correct limits of integration.
- The second point can be earned without the first point. The second point is earned for the presentation of a correct antiderivative of a function of the form $Ax\sqrt{4 - x^2}$, for any nonzero constant A. If the response has subsequent errors in simplification of the antiderivative or sign errors, the response will earn the second point but will not earn the third point.
- Responses that use u-substitution and have incorrect limits of integration or do not change the limits of integration from x- to u-values are eligible for the second point.
- The response is eligible for the third point only if it has earned the second point.
- The third point is earned only for the answer 16 or equivalent. In the case where a response only presents an indefinite integral, the use of the correct limits of integration to evaluate the antiderivative must be shown to earn the third point.
- The response cannot correct -16 to $+16$ in order to earn the third point; there is no possible reversal here.

Total for part (a) 3 points

(b) It is known that, for $y = cx\sqrt{4 - x^2}$, $\dfrac{dy}{dx} = \dfrac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}}$. For a particular spinning toy, the radius of the largest cross-sectional circular slice is 1.2 inches. What is the value of c for this spinning toy?

The cross-sectional circular slice with the largest radius occurs where $cx\sqrt{4 - x^2}$ has its maximum on the interval $0 < x < 2$. $\dfrac{dy}{dx} = \dfrac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}} = 0 \Rightarrow x = \sqrt{2}$	Sets $\dfrac{dy}{dx} = 0$	**1 point**
$x = \sqrt{2} \Rightarrow y = c\sqrt{2}\sqrt{4 - \left(\sqrt{2}\right)^2} = 2c$ $2c = 1.2 \Rightarrow c = 0.6$	Answer	**1 point**

Scoring notes:

- The first point is earned for setting $\dfrac{dy}{dx} = 0$, $\dfrac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}} = 0$, or $c\left(4 - 2x^2\right) = 0$.
- An unsupported $x = \sqrt{2}$ does not earn the first point.
- The second point can be earned without the first point but is earned only for the answer $c = 0.6$ with supporting work.

Total for part (b) 2 points

(c) For another spinning toy, the volume is 2π cubic inches. What is the value of c for this spinning toy?

Volume $= \int_0^2 \pi\left(cx\sqrt{4-x^2}\right)^2 dx = \pi c^2 \int_0^2 x^2\left(4-x^2\right) dx$	Form of the integrand	**1 point**
	Limits and constant	**1 point**
$= \pi c^2 \int_0^2 \left(4x^2 - x^4\right) dx = \pi c^2 \left(\frac{4}{3}x^3 - \frac{1}{5}x^5 \Big\|_0^2\right)$	Antiderivative	**1 point**
$= \pi c^2\left(\frac{32}{3} - \frac{32}{5}\right) = \frac{64\pi c^2}{15}$ $\frac{64\pi c^2}{15} = 2\pi \Rightarrow c^2 = \frac{15}{32} \Rightarrow c = \sqrt{\frac{15}{32}}$	Answer	**1 point**

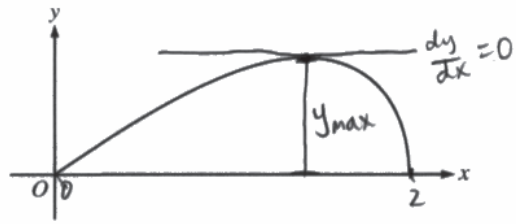
Scoring notes:

- The first point is earned for presenting an integrand of the form $A\left(x\sqrt{4-x^2}\right)^2$ in a definite integral with any limits of integration (numeric or alphanumeric) and any nonzero constant A. Mishandling the c will result in the response being ineligible for the fourth point.
- The second point can be earned without the first point. The second point is earned for the limits of integration, $x = 0$ and $x = 2$, and the constant π (but not for 2π) as part of an integral with a correct or incorrect integrand.
- If an indefinite integral is presented with the correct constant π, the second point can be earned if the antiderivative (correct or incorrect) is evaluated using the correct limits of integration.
- A response that presents $2 = \int_0^2 \left(cx\sqrt{4-x^2}\right)^2 dx$ earns the first and second points.
- The third point is earned for presenting a correct antiderivative of the presented integrand of the form $A\left(x\sqrt{4-x^2}\right)^2$ for any nonzero A. If there are subsequent errors in simplification of the antiderivative, linkage errors, or sign errors, the response will not earn the fourth point.
- The fourth point cannot be earned without the third point. The fourth point is earned only for the correct answer. The expression does not need to be simplified to earn the fourth point.

Total for part (c) 4 points

Total for question 3 9 points

Answer QUESTION 3 part (a) on this page.



Response for question 3(a)

$y = 6x\sqrt{4-x^2} = 0$

$x = 0 \;,\; x = 2$

$A = \int_0^2 6x\sqrt{4-x^2}\,dx \qquad u = 4-x^2$

$du = -2x\,dx$

$A = \int_4^0 -3\sqrt{u}\,du = 3\int_0^4 u^{\frac{1}{2}}\,du = 3\,u^{3/2}\cdot\frac{2}{3}\Big|_0^4$

$A = 2(4^{3/2} - 0^{3/2}) = 2(2^{2\cdot 3/2} - 0) = 2(8) = 16$

$\boxed{A = 16}$

3 3 3 3 3 NO CALCULATOR ALLOWED 3 3 3 3 3

Answer QUESTION 3 parts (b) and (c) on this page.

Response for question 3(b)

Largest cross section when y is greatest (maximum of y on graph).

Find max:

$\frac{dy}{dx} = \frac{c(4-2x^2)}{\sqrt{4-x^2}} = 0 \rightarrow c(4-2x^2)=0$

$4 = 2x^2$

$x = \sqrt{2}$

At $x=\sqrt{2}$, $y=1.2$ (largest radius of cross-section equals 1.2, which is max y value)

$y = cx\sqrt{4-x^2}$

$1.2 = c\sqrt{2}(\sqrt{4-(\sqrt{2})^2}) = c\sqrt{2}(\sqrt{4-2}) = c\sqrt{2}(\sqrt{2}) = 2c$

$c = \frac{1.2}{2} = 0.6 \rightarrow \boxed{c = 0.6}$

Response for question 3(c)

$V = \pi\int_0^2 y^2\,dx = \pi\int_0^2 (cx\sqrt{4-x^2})^2\,dx = \pi c^2\int_0^2 x^2(4-x^2)\,dx = \pi c^2\int_0^2 (4x^2-x^4)\,dx$

$V = \pi c^2\left[\frac{4x^3}{3} - \frac{x^5}{5}\Big|_0^2\right] = \pi c^2\left[\left(\frac{4(8)}{3} - \frac{(32)}{5}\right) - (0-0)\right]$

$V = \pi c^2\left(\frac{32(5)}{3(5)} - \frac{32(3)}{5(3)}\right) = \pi c^2\left(\frac{2(32)}{15}\right) = \pi c^2\left(\frac{64}{15}\right)$

$2\pi = \pi c^2\left(\frac{64}{15}\right)$

$c^2 = \frac{30}{64} \rightarrow c = \sqrt{\frac{30}{64}} = \boxed{\frac{\sqrt{30}}{8}}$

3 3 3 3 3 NO CALCULATOR ALLOWED 3 3 3 3 3

Answer QUESTION 3 part (a) on this page.

Response for question 3(a)

$y = 6x\sqrt{4-x^2}$

$6x\sqrt{4-x^2} = 0$

$x = 0$

$x = 2$

$A = \int_0^2 6x\sqrt{4-x^2}\,dx$

$u = 4-x^2$

$du = -2x\,dx$

$-3du = 6x\,dx$

$u = 4-2^2$ $\quad u = 4-0^2$

$u = 0$ $\quad u = 4$

$A = \int_4^0 -3\sqrt{u}\,du$

$\frac{1}{2} + \frac{2}{2} = \frac{3}{2}$

$A = -3\int_4^0 \sqrt{u}\,du$

$-3\left[\frac{2(u)^{3/2}}{3}\right]\Big|_4^0$

$-3\left[\frac{2(4-x^2)^{3/2}}{3}\right]\Big|_0^2$

$= -3\left[\frac{2(4-2^2)^{3/2}}{3} - \frac{2(4-0^2)^{3/2}}{3}\right]$

$= -3\left(0 - \frac{2(4)^{3/2}}{3}\right)$

$= \frac{6(4)^{3/2}}{3} = 2(4)^{3/2}$

$= 16$

3 3 3 3 3 NO CALCULATOR ALLOWED 3 3 3 3 3

Answer QUESTION 3 parts (b) and (c) on this page.

Response for question 3(b)

$\frac{dy}{dx} = 0$

$\frac{c(4-2x^2)}{\sqrt{4-x^2}} = 0$

$c(4-2x^2) = 0$

$4-2x^2 = 0$

$4 = 2x^2$

$2 = x^2$

$x = \sqrt{2}$

$1.2 = c\sqrt{2} \times \sqrt{4-(\sqrt{2})^2}$

$= c\sqrt{2} \times \sqrt{4-2}$

$= c\sqrt{2} \times \sqrt{2}$

$1.2 = 2c$

$c = \frac{1.2}{2}$

$c = 0.6$

Response for question 3(c)

$V = \pi \int r^2 dx$

$V = \pi \int_0^2 cx\sqrt{4-x^2}\, dx$

$c\pi \int_0^2 x\sqrt{4-x^2}\, dx = 2\pi$

$c\pi \int \sqrt{u}\, du = 2\pi$

$c\pi \left[\frac{2u^{3/2}}{3} \Big|_4^0 \right] = 2\pi$

$u = 4-x^2$

$\frac{du}{-2} = \frac{-2x}{-2} dx$

$du = x\, dx$

$u = 4-2^2 = 0$

$4-0^2 = 4$

$\frac{2(4-x^2)^{3/2}}{3} \Big|_0^2 = \frac{2}{c}$

$0 - \frac{2(4)^{3/2}}{3} = \frac{2}{c}$

$\frac{-16}{3} = \frac{2}{c}$

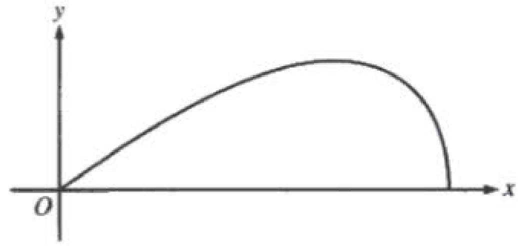
$c = \frac{2 \times 3}{-16}$

$c = \frac{-6}{16}$

$c = \frac{-3}{8}$

3 3 3 3 3 NO CALCULATOR ALLOWED 3 3 3 3 3

Answer QUESTION 3 part (a) on this page.



Response for question 3(a)

$y = 6x\sqrt{4-x^2}$ $\quad y = 0 = 6x\sqrt{4-x^2}$

$x = 0, 2$

$A = \int_0^2 6x(4-x^2)^{1/2}\,dx$

$A = \frac{1}{2}6\int_0^2 (2x)(4-x^2)^{1/2}\,dx$

$A = 3\left[\frac{2}{3}(4-x^2)^{3/2}\right)\Big|_0^2$

$A = 3\left(\frac{2}{3}(0) - \frac{2}{3}(4)^{3/2}\right)$

$3\left(0 - \frac{16}{3}\right)$

$|-16| = \boxed{16 \text{ inches}^2}$

3 3 3 3 3 NO CALCULATOR ALLOWED 3 3 3 3 3

Answer QUESTION 3 parts (b) and (c) on this page.

Response for question 3(b)

$y = Cx\sqrt{4-x^2}$ $\qquad \frac{dy}{dx} = \frac{C(4-2x^2)}{\sqrt{4-x^2}} = 0$

$4 = 2x^2$

$2 = x^2$

$x = \sqrt{2}$ CN $\qquad \frac{dy}{dx}$: +++++++ | - - - - at $\sqrt{2}$

$y(\sqrt{2}) = 1.2 = C\sqrt{2}\sqrt{4-2}$

$1.2 = C\sqrt{2}\sqrt{2}$

$1.2 = 2C$

$\boxed{C = .6}$

Response for question 3(c)

$y = 6x\sqrt{4-x^2} = 0$

$x = 0, 2$

$V = 2\pi \int_0^2 x\left(Cx\sqrt{4-x^2}\right) dx$

$V = 2\pi = 2\pi \int_0^2 x\left(Cx\sqrt{4-x^2}\right) dx$

$\int_0^2 x\left(Cx\sqrt{4-x^2}\right) dx = k$

?

Question 3

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem a company designs spinning toys using various functions of the form $y = cx\sqrt{4 - x^2}$, where c is a positive constant. A graph of the region in the first quadrant bounded by the x-axis and this function for some c is given and students were told that the spinning toys are in the shape of the solid generated when this region is revolved around the x-axis. Both x and y are measured in inches.
In part (a) students were asked to find the area of the region in the first quadrant bounded by the x-axis and the region $y = cx\sqrt{4 - x^2}$ for $c = 6.$ A correct response will set up the definite integral $\int_0^2 6x\sqrt{4 - x^2}\,dx$ and use the method of substitution to evaluate the integral to obtain an area of 16.
In part (b) students were told that for $y = cx\sqrt{4 - x^2}$, $\frac{dy}{dx} = \frac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}}$. They were also told that for a particular spinning toy the radius of the largest cross-sectional circular slice is 1.2 inches and were asked to find the value of c for this particular spinning toy. A correct response will solve $\frac{dy}{dx} = 0$ to find that the largest radius occurs when $x = \sqrt{2}$. Then using this value of x in the equation $y = cx\sqrt{4 - x^2} = 1.2,$ the value of c is found to be 0.6.
In part (c) students were told that for another spinning toy, the volume is 2π cubic inches. They were asked to find the value of c for this spinning toy. A correct response would set up the volume of the toy as the integral $\int_0^2 \pi\left(cx\sqrt{4 - x^2}\right)^2 dx,$ evaluate this integral, and set the value equal to 2π. Solving the resulting equation for c results in $c = \sqrt{\frac{15}{32}}$.

Sample: 3A
Score: 9

The response earned 9 points: 3 points in part (a), 2 points in part (b), and 4 points in part (c). In part (a) the response presents the correct integrand in a definite integral and earned the first point. The antiderivative of $3u^{3/2} \cdot \frac{2}{3}$ with the definition $u = 4 - x^2$ is correct and earned the second point. The response has the correct answer and earned the third point. In part (b) the response earned the first point for stating $\frac{dy}{dx} = \frac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}} = 0.$ The answer is correct, and the response earned the second point. In part (c) the response presents y^2 as the integrand of a definite integral and earned the first point. Note that because $y = cx\sqrt{4 - x^2}$ is given in the statement of the problem, a response can reference the function by using y for the first point. The limits and constant are correct and earned the second point. The antiderivative is correct and earned the third point. The response is eligible for the fourth point. The answer is correct and earned the fourth point. Note that $\frac{\sqrt{30}}{8} = \sqrt{\frac{15}{32}}$.

Question 3 (continued)

Sample: 3B
Score: 6

The response earned 6 points: 3 points in part (a), 2 points in part (b), and 1 point in part (c). In part (a) the response presents the correct integrand in a definite integral and earned the first point. The antiderivative $-3\left[\frac{2(u)^{3/2}}{3}\right]$ with the definition $u = 4 - x^2$ is correct and earned the second point. Note that the sign of the antiderivative is consistent with the limits of integration. The response is eligible for the third point. The answer is correct and earned the third point. Note that the substitution of $u = 4 - x^2$ after finding the antiderivative and using the limits of $x = 0$ and $x = 2$ is not necessary to evaluate the antiderivative. In part (b) the response earned the first point by stating $\frac{dy}{dx} = 0.$ The answer is correct and earned the second point. In part (c) the integrand is not of the correct form and the response did not earn the first point. The limits and constant are correct and earned the second point. Because the integrand is not of the correct form, the response is not eligible for and did not earn the third point. Without earning the third point, the response is not eligible for and did not earn the fourth point.

Sample: 3C
Score: 3

The response earned 3 points: 1 point in part (a), 2 points in part (b), and no points in part (c). In part (a) the response presents the correct integrand in a definite integral and earned the first point. The antiderivative presented is incorrect because the sign of the antiderivative is incorrect, and the response did not earn the second point. The response is not eligible for and did not earn the third point. In part (b) the response earned the first point by stating $\frac{dy}{dx} = \frac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}} = 0.$ The answer is correct, and the response earned the second point. In part (c) the integrand presented is not of the correct form and the response did not earn the first point. The constant 2π is incorrect and the response did not earn the second point. Without an integrand of the correct form, the response is not eligible for the third point and is not eligible for the fourth point. The response did not earn the third point and did not earn the fourth point.