
AP[®] Calculus AB

Sample Student Responses and Scoring Commentary

Inside:

Free Response Question 5

- Scoring Guideline**
- Student Samples**
- Scoring Commentary**

Part B (AB): Graphing calculator not allowed**Question 5****9 points****General Scoring Notes**

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.

Consider the function $y = f(x)$ whose curve is given by the equation $2y^2 - 6 = y \sin x$ for $y > 0$.

	Model Solution	Scoring
(a)	Show that $\frac{dy}{dx} = \frac{y \cos x}{4y - \sin x}$.	
	$\frac{d}{dx}(2y^2 - 6) = \frac{d}{dx}(y \sin x) \Rightarrow 4y \frac{dy}{dx} = \frac{dy}{dx} \sin x + y \cos x$	Implicit differentiation 1 point
	$\Rightarrow 4y \frac{dy}{dx} - \frac{dy}{dx} \sin x = y \cos x \Rightarrow \frac{dy}{dx}(4y - \sin x) = y \cos x$ $\Rightarrow \frac{dy}{dx} = \frac{y \cos x}{4y - \sin x}$	Verification 1 point

Scoring notes:

- The first point is earned only for correctly implicitly differentiating $2y^2 - 6 = y \sin x$. Responses may use alternative notations for $\frac{dy}{dx}$, such as y' .
- The second point may not be earned without the first point.
- It is sufficient to present $\frac{dy}{dx}(4y - \sin x) = y \cos x$ to earn the second point, provided that there are no subsequent errors.

Total for part (a) 2 points

- (b) Write an equation for the line tangent to the curve at the point $(0, \sqrt{3})$.

$$\text{At the point } (0, \sqrt{3}), \frac{dy}{dx} = \frac{\sqrt{3} \cos 0}{4\sqrt{3} - \sin 0} = \frac{1}{4}.$$

$$\text{An equation for the tangent line is } y = \sqrt{3} + \frac{1}{4}x.$$

Answer

1 point**Scoring notes:**

- Any correct tangent line equation will earn the point. No supporting work is required. Simplification of the slope value is not required.

Total for part (b) 1 point

- (c) For $0 \leq x \leq \pi$ and $y > 0$, find the coordinates of the point where the line tangent to the curve is horizontal.

$$\frac{dy}{dx} = \frac{y \cos x}{4y - \sin x} = 0 \Rightarrow y \cos x = 0 \text{ and } 4y - \sin x \neq 0$$

Sets $\frac{dy}{dx} = 0$

1 point

$$y \cos x = 0 \text{ and } y > 0 \Rightarrow x = \frac{\pi}{2}$$

$x = \frac{\pi}{2}$

1 point

$$\text{When } x = \frac{\pi}{2}, y \sin x = 2y^2 - 6 \Rightarrow y \sin \frac{\pi}{2} = 2y^2 - 6$$

$$\Rightarrow y = 2y^2 - 6 \Rightarrow 2y^2 - y - 6 = 0$$

$$\Rightarrow (2y + 3)(y - 2) = 0 \Rightarrow y = 2$$

$y = 2$

1 point

When $x = \frac{\pi}{2}$ and $y = 2$, $4y - \sin x = 8 - 1 \neq 0$. Therefore, the

line tangent to the curve is horizontal at the point $\left(\frac{\pi}{2}, 2\right)$.

Scoring notes:

- The first point is earned by any of $\frac{dy}{dx} = 0$, $\frac{y \cos x}{4y - \sin x} = 0$, $y \cos x = 0$, or $\cos x = 0$.
- If additional “correct” x -values are considered outside of the given domain, the response must commit to only $x = \frac{\pi}{2}$ to earn the second point. Any presented y -values, correct or incorrect, are not considered for the second point.
- Entering with $x = \frac{\pi}{2}$ does not earn the first point, earns the second point, and is eligible for the third point. The third point is earned for finding $y = 2$. The coordinates do not have to be presented as an ordered pair.
- The third point is not earned with additional points present unless the response commits to the correct point.

Total for part (c) 3 points

- (d) Determine whether f has a relative minimum, a relative maximum, or neither at the point found in part (c). Justify your answer.

$\frac{d^2y}{dx^2} = \frac{(4y - \sin x)\left(\frac{dy}{dx} \cos x - y \sin x\right) - (y \cos x)\left(4\frac{dy}{dx} - \cos x\right)}{(4y - \sin x)^2}$	Considers $\frac{d^2y}{dx^2}$ 1 point
<p>When $x = \frac{\pi}{2}$ and $y = 2$,</p> $\frac{d^2y}{dx^2} = \frac{(4 \cdot 2 - \sin \frac{\pi}{2})\left(0 \cdot \cos \frac{\pi}{2} - 2 \cdot \sin \frac{\pi}{2}\right) - \left(2 \cos \frac{\pi}{2}\right)\left(4 \cdot 0 - \cos \frac{\pi}{2}\right)}{(4 \cdot 2 - \sin \frac{\pi}{2})^2}$ $= \frac{(7)(-2) - (0)(0)}{(7)^2} = \frac{-2}{7} < 0.$	$\frac{d^2y}{dx^2}$ at $\left(\frac{\pi}{2}, 2\right)$ 1 point
<p>f has a relative maximum at the point $\left(\frac{\pi}{2}, 2\right)$ because $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$.</p>	Answer with justification 1 point

Scoring notes:

- The first point is earned for an attempt to use the quotient rule (or product rule) to find $\frac{d^2y}{dx^2}$.
- The second point is earned for correctly finding $\frac{d^2y}{dx^2}$ and evaluating to find that $\frac{d^2y}{dx^2} < 0$ at $\left(\frac{\pi}{2}, 2\right)$. The explicit value of $-\frac{2}{7}$ or the equivalent does not need to be reported, but any reported values must be correct in order to earn this point.
- The third point can be earned without the second point by reaching a consistent conclusion based on the reported sign of a nonzero value of $\frac{d^2y}{dx^2}$ obtained utilizing $\frac{dy}{dx} = 0$.
- Imports: A response is eligible to earn all 3 points in part (d) with a point of the form $\left(\frac{\pi}{2}, k\right)$ with $k > 0$, imported from part (c).

Alternate Solution for part (d)	Scoring for Alternate Solution	
For the function $y = f(x)$ near the point $\left(\frac{\pi}{2}, 2\right)$, $4y - \sin x > 0$ and $y > 0$.	Considers sign of $4y - \sin x$	1 point
Thus, $\frac{dy}{dx} = \frac{y \cos x}{4y - \sin x}$ changes from positive to negative at $x = \frac{\pi}{2}$.	$\frac{dy}{dx}$ changes from positive to negative at $x = \frac{\pi}{2}$	1 point
By the First Derivative Test, f has a relative maximum at the point $\left(\frac{\pi}{2}, 2\right)$.	Conclusion	1 point

Scoring notes:

- The first point for considering the sign of $4y - \sin x$ may also be earned by stating that $4y - \sin x$ is not equal to zero.
- The second and third points can be earned without the first point.
- To earn the second point a response must state that $\frac{dy}{dx}$ (or $\cos x$) changes from positive to negative at $x = \frac{\pi}{2}$.
- The third point cannot be earned without the second point.
- A response that concludes there is a minimum at this point does not earn the third point.

Total for part (d) 3 points

Total for question 5 9 points

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (a) and (b) on this page.

Response for question 5(a)

$$2y^2 - 6 = y \sin x$$

$$4yy' = y \cos x + y' \sin x$$

$$4yy' - y' \sin x = y \cos x$$

$$y'(4y - \sin x) = y \cos x$$

$$y' = \frac{y \cos x}{4y - \sin x}$$

$$\frac{dy}{dx} = \frac{y \cos x}{4y - \sin x}$$

Response for question 5(b)

$$b. \frac{dy}{dx} \text{ at } (0, \sqrt{3}) = \frac{\sqrt{3} \cos(0)}{4\sqrt{3} - \sin(0)} = \frac{\sqrt{3}(1)}{4\sqrt{3} - 0} = \frac{\sqrt{3}}{4\sqrt{3}} = \frac{1}{4}$$

$$y - \sqrt{3} = \frac{1}{4}(x - 0)$$

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (c) and (d) on this page.

Response for question 5(c)

$$\frac{dy}{dx} = \frac{y \cos x}{4y - \sin x}$$

$$0 = \frac{y \cos x}{4y - \sin x}$$

$y \neq 0$ $x \neq 0$

$$0 = y \cos x$$

since $\cos(0) = 1$ and $0 \neq 1$

$$x = \frac{\pi}{2}$$

$$2y^2 - 6 = y \sin x$$

$$y \neq 0: 2y^2 - 6 = 0 \sin x$$

$$y \neq 0$$

$$x = \frac{\pi}{2}: 2y^2 - 6 = y \sin\left(\frac{\pi}{2}\right)$$

$$2y^2 - 6 = y(1)$$

$$2y^2 - y - 6 = 0$$

$$(2y+3)(y-2) = 0$$

$$y = 2 \quad 2y+3 = 0$$

$$2y = -3$$

$$y = -\frac{3}{2}$$

$f(x)$ has a horizontal tangent at $\left(\frac{\pi}{2}, 2\right)$ and $\left(\frac{\pi}{2}, -\frac{3}{2}\right)$

Response for question 5(d)

$$\frac{dy}{dx} = \frac{y \cos x}{4y - \sin x}$$

$$\frac{d^2y}{dx^2} = \frac{(4y - \sin x)[y(-\sin x) + y' \cos x] - (y \cos x)(4y' - \cos x)}{(4y - \sin x)^2}$$

$$\text{At } \left(\frac{\pi}{2}, 2\right): \frac{d^2y}{dx^2} = \frac{(4 \cdot 2 - 1)[2(-1) + 0 \cos x] - (2 \cos x)(4(0) - 0)}{(4(2) - 1)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(8-1)(-2)}{(8-1)^2} = \frac{7(-2)}{7^2} = -\frac{2}{7}$$

Concave down

Since $\frac{d^2y}{dx^2} < 0$ at $\left(\frac{\pi}{2}, 2\right)$, f has a relative maximum at that point.

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (a) and (b) on this page.

Response for question 5(a)

$$\begin{aligned}
 2y^2 - 6 &= (y \sin x) \\
 4y \frac{dy}{dx} &= (y \cos x) + (\sin x \frac{dy}{dx}) \\
 (4y \frac{dy}{dx} - \sin x \frac{dy}{dx}) &= y \cos x \\
 \frac{dy}{dx} (4y - \sin x) &= y \cos x \\
 \frac{dy}{dx} &= \frac{y \cos x}{4y - \sin x}
 \end{aligned}$$

Response for question 5(b)

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{y \cos x}{4y \sin x} \Big|_{(0, \sqrt{3})} = \frac{3 \cos(0)}{4(\sqrt{3} \sin(0))} = \frac{\sqrt{3}}{4} \\
 y - \sqrt{3} &= \frac{\sqrt{3} \cos(0)}{4(\sqrt{3} \sin(0))} (x - 0)
 \end{aligned}$$

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (c) and (d) on this page.

Response for question 5(c)

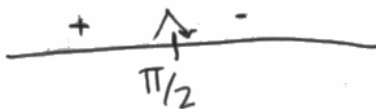
$$\frac{dy}{dx} = \frac{y \cos x}{4y - \sin x} = 0$$

$$y \cos x = 0$$

$$\cos(\pi/2) = 0$$

at the point $(\pi/2, 1)$ the line tangent to the curve is horizontal

Response for question 5(d)



On the point $(\pi/2, 1)$ f has a relative maximum because the values of $f'(x)$ switch from positive to negative at this x -value.

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (a) and (b) on this page.

Response for question 5(a)

$$4y \frac{dy}{dx} = \frac{dy}{dx} \sin x + y \cos x$$

$$4y \frac{dy}{dx} - \sin x \frac{dy}{dx} = y \cos x$$

$$\frac{dy}{dx} = \frac{y \cos x}{4y - \sin x}$$

Response for question 5(b)

$$2(3) - 6 = \sqrt{3} \sin 0$$

$$6 - 6 = 0$$

$$y - \sqrt{3} = \frac{y \cos x}{4y - \sin x} (x)$$

$$y = \frac{y \cos x}{4y - \sin x} (x) + \sqrt{3}$$

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (c) and (d) on this page.

Response for question 5(c)

$$2y^2 - 6 = y$$

$$x = \pi$$

$$2y^2 - 6 = y \quad (1)$$

$$2y^2 - 6y = 0$$

$$2(y^2 - 3) = 0 \implies y = \pm\sqrt{3}$$

$$\frac{dy}{dx} = \frac{y \cos \pi}{4y - \sin \pi} = 0$$

~~$(0, \sqrt{3})$~~ $x = \pi$

(π, \dots)

$$\frac{dy}{dx} = 0 \implies \frac{y \cos x}{4y - \sin x}$$

Response for question 5(d)

$$\frac{d^2y}{dx^2} = \frac{(4y - \sin x) \left(\cos \pi \frac{dy}{dx} \right) (-y \sin x) - (4 \frac{dy}{dx} + \cos x) (y \cos \pi)}{(4y - \sin x)^2}$$

if $\frac{d^2y}{dx^2} > 0$

Question 5

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem $y = f(x)$ is an implicitly defined function whose curve is given by $2y^2 - 6 = y \sin x$ for $y > 0$.

In part (a) students were asked to show that $\frac{dy}{dx} = \frac{y \cos x}{4y - \sin x}$, which can be done using implicit differentiation.

In part (b) students were asked to write an equation for the tangent line at the point $(0, \sqrt{3})$. A correct response would evaluate the derivative given in part (a) at the point $(0, \sqrt{3})$ and then write the equation of a line through the given point with slope equated to the evaluated derivative.

In part (c) students were asked to find the coordinates of the point where the line tangent to the curve is horizontal for $0 \leq x \leq \pi$ and $y > 0$. A correct response would set the slope of the tangent line, $\frac{dy}{dx}$, equal to zero, then

determine that $y \cos x = 0$ when $x = \frac{\pi}{2}$. The response should then use the given equation $2y^2 - 6 = y \sin x$ to find $y = 2$ when $x = \frac{\pi}{2}$, which results in the point with coordinates $\left(\frac{\pi}{2}, 2\right)$.

In part (d) students were asked to determine and justify whether the function f has a relative minimum, a relative maximum, or neither at the point found in part (c): $\left(\frac{\pi}{2}, 2\right)$. A correct response would use the quotient rule to find

$\frac{d^2y}{dx^2}$, determine the sign of $\frac{d^2y}{dx^2}$ at the critical point $\left(\frac{\pi}{2}, 2\right)$, and conclude that f has a relative maximum at this point.

Sample: 5A

Score: 9

The response earned 9 points: 2 points in part (a), 1 point in part (b), 3 points in part (c), and 3 points in part (d). In part (a) the response earned the first point in line 2 with a correct implicit differentiation equation. Having earned the first point, the response is eligible to earn the second point. The response earned the second point with correct

algebraic work in lines 3, 4, 5, and 6, verifying the given expression for $\frac{dy}{dx}$. Note that the response would have

earned the second point with either line 3 or line 4 leading to either line 5 or line 6. In part (b) the response earned the point for a correct equation of the tangent line on line 2. In part (c) the response earned the first point at the

beginning of line 2 for setting the given expression for $\frac{dy}{dx}$ equal to 0. The response would have earned the second

point at the beginning of line 6 for the equation $x = \frac{\pi}{2}$ with no other x -values present. In this case, the response

earned the second and third points with the commitment to the single ordered pair $\left(\frac{\pi}{2}, 2\right)$ in the circled statement.

In part (d) the response earned the first point in line 2 for an attempt to find $\frac{d^2y}{dx^2}$ using the quotient rule. The

response earned the second point for a correct expression for $\frac{d^2y}{dx^2}$ found on line 2 followed by a correct evaluation

Question 5 (continued)

of $\frac{d^2y}{dx^2}$ at the point $\left(\frac{\pi}{2}, 2\right)$ in line 3 with no subsequent errors. The response earned the third point with the circled statement, presenting a correct conclusion with the justification “ $\frac{d^2y}{dx^2} < 0$ at $\left(\frac{\pi}{2}, 2\right)$.”

Sample: 5B**Score: 6**

The response earned 6 points: 2 points in part (a), no points in part (b), 2 points in part (c), and 2 points in part (d). In part (a) the response earned the first point in line 2 with a correct implicit differentiation equation. Having earned the first point, the response is eligible to earn the second point. The response would have earned the second point with the work in either line 3 or line 4 leading to line 5. In this case, the response earned the second point with correct algebraic verification work in lines 3, 4, and 5. In part (b) the response did not earn the point because there is an error in the presentation of the slope value, missing the subtraction in the denominator of the expression. In part (c) the response earned the first point in line 1 by setting $\frac{dy}{dx}$ equal to 0. The response earned the second point in line 4 with the correct x -value of $\frac{\pi}{2}$ presented in the ordered pair. The response presents an incorrect y -value of 1 in the ordered pair and did not earn the third point. In part (d) the response does not present an attempt to find the second derivative as required in the primary solution shown in the scoring guide, so the alternate solution is considered. The response does not reference the sign of $4y - \sin x$ and did not earn the first point. The response is eligible for the second and third points because the response references a point with the correct x -value of $\frac{\pi}{2}$. The response earned the second and third points with the statement “ f has a relative maximum because the values of $f'(x)$ switch from positive to negative at this x -value.” Note that the stem of the question states that $y = f(x)$, thus $f'(x)$ is an acceptable alternative notation for $\frac{dy}{dx}$.

Sample: 5C**Score: 4**

The response earned 4 points: 2 points in part (a), no points in part (b), 1 point in part (c), and 1 point in part (d). In part (a) the response earned the first point in line 1 for a correct implicit differentiation of the given equation. Having earned the first point, the response is eligible to earn the second point. The response earned the second point with correct algebraic work in lines 2 and 3. This response demonstrates a minimum amount of verification work required to earn the second point. In part (b) the response did not earn the point. The response does not present a correct numerical expression for the slope in the equation of the tangent line. In part (c) the response earned the first point on the last line for the equation $\frac{dy}{dx} = 0$. The response does not present the correct x -value, so did not earn the second point. The response does not present a y -coordinate and so did not earn the third point. In part (d) the response earned the first point for an attempt at finding $\frac{d^2y}{dx^2}$ using the quotient rule. The attempt contains errors, so the response is not eligible for the second point. The response presents no further work leading to a consistent conclusion, so the response did not earn the third point.