2021

AP CollegeBoard

AP® Calculus AB

Sample Student Responses and Scoring Commentary

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Free Response Question 6

☑ Scoring Guideline

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Part B (AB): Graphing calculator not allowed
Question 6 — 9 points

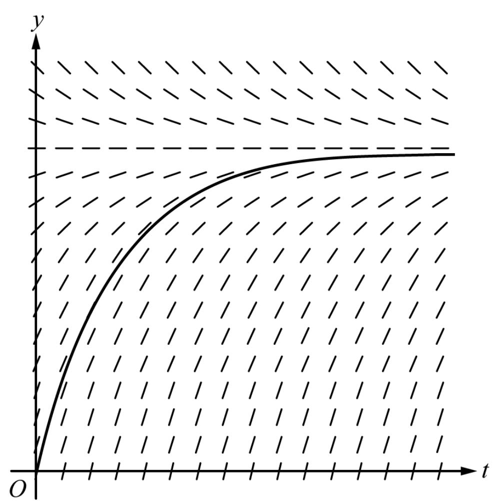
General Scoring Notes
Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.

A medication is administered to a patient. The amount, in milligrams, of the medication in the patient at time t hours is modeled by a function $y = A(t)$ that satisfies the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$. At time $t = 0$ hours, there are 0 milligrams of the medication in the patient.

Model Solution	Scoring	

(a) A portion of the slope field for the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$ is given below. Sketch the solution curve through the point $(0, 0)$.



Solution curve **1 point**

Scoring notes:

- To earn the point the solution curve must pass through the point $(0, 0)$, be generally increasing and concave down, and approach the horizontal asymptote from below as t increases. The point is not earned if two or more solution curves are presented.

Total for part (a) 1 point

(b) Using correct units, interpret the statement $\lim_{t\to\infty} A(t) = 12$ in the context of this problem.

Over time the amount of medication in the patient approaches 12 milligrams.	Interpretation	**1 point**

Scoring notes:

- To earn the point the interpretation must include "medication in the patient," "approaches 12," and units (milligrams), or their equivalents.

Total for part (b) **1 point**

(c) Use separation of variables to find $y = A(t)$, the particular solution to the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$ with initial condition $A(0) = 0$.

$\frac{dy}{dt} = \frac{12 - y}{3} \Rightarrow \frac{dy}{12 - y} = \frac{dt}{3}$	Separation of variables	**1 point**
$\int \frac{dy}{12 - y} = \int \frac{dt}{3} \Rightarrow -\ln\lvert 12 - y\rvert = \frac{t}{3} + C$	Antiderivatives	**1 point**
$\ln\lvert 12 - y\rvert = -\frac{t}{3} - C \Rightarrow \lvert 12 - y\rvert = e^{-t/3-C}$ $\Rightarrow y = 12 + Ke^{-t/3}$ $0 = 12 + K \Rightarrow K = -12$	Constant of integration and uses initial condition	**1 point**
$y = A(t) = 12 - 12e^{-t/3}$	Solves for y	**1 point**

Scoring notes:

- A response of $\frac{dy}{12 - y} = 3\,dt$ is a bad separation and does not earn the first point. However, this response is eligible for the second and third points. It cannot earn the fourth point.
- Absolute value bars are not required in this part.
- A response that correctly separates to $\frac{3\,dy}{12 - y} = dt$ but then incorrectly simplifies to $\frac{dy}{4 - y} = dt$ earns the first point (for the initial correct separation), is eligible for the second point (for $-\ln\lvert 4 - y\rvert = t$, with or without $+C$), but is not eligible for the third or fourth points.
- $+\ln\lvert 12 - y\rvert = \frac{t}{3}$ (with or without $+C$) does not earn the second point and is not eligible for the fourth point; $+\ln\lvert 12 - y\rvert = \frac{t}{3} + C$ is eligible for the third point.
- In all other cases, the points are earned consecutively—the second point cannot be earned without the first, the third without the second, etc.

Total for part (c) **4 points**

(d) A different procedure is used to administer the medication to a second patient. The amount, in milligrams, of the medication in the second patient at time t hours is modeled by a function $y = B(t)$ that satisfies the differential equation $\frac{dy}{dt} = 3 - \frac{y}{t+2}$. At time $t = 1$ hour, there are 2.5 milligrams of the medication in the second patient. Is the rate of change of the amount of medication in the second patient increasing or decreasing at time $t = 1$? Give a reason for your answer.

$\frac{dy}{dt} = 3 - \frac{y}{t+2} \Rightarrow \frac{d^2y}{dt^2} = (-1)\frac{\frac{dy}{dt}(t+2) - y}{(t+2)^2}$	Quotient rule	**1 point**
$B'(1) = 3 - \frac{B(1)}{3} = 3 - \frac{2.5}{3} = \frac{6.5}{3}$ $B''(1) = -\frac{B'(1)\cdot 3 - B(1)}{3^2} = -\frac{6.5 - 2.5}{9} = -\frac{4}{9} < 0$	$B''(1) < 0$	**1 point**
The rate of change of the amount of medication is decreasing at time $t = 1$ because $B''(1) < 0$ and $\frac{d^2y}{dt^2}$ is continuous in an interval containing $t = 1$.	Answer with reason	**1 point**

Scoring notes:

- The first point is for correctly applying the quotient rule to $\frac{y}{t+2}$ or applying the product rule to $y(t+2)^{-1}$. Errors in differentiating the constant, 3, or handling the sign of the second term of $\frac{dy}{dt}$ will result in not earning the second point.
- The second point cannot be earned unless the second derivative $\frac{d^2y}{dt^2}$ is correct.
- For the second point it is sufficient to state the sign of $B''(1)$ is negative with supporting work. If a value is declared for $B''(1)$, it must be correct in order to earn the second point.
- Eligibility for the third point: An attempt at using the quotient rule (or product rule) to find $B''(1)$. In this case the third point will be earned for a consistent conclusion based on the declared value (or sign) of $B''(1)$.

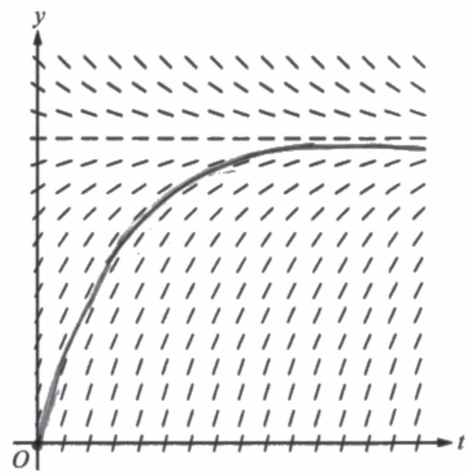
Total for part (d)　3 points

Total for question 6　9 points

6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 parts (a) and (b) on this page.

Response for question 6(a)



Response for question 6(b)

As time goes to infinity, the amount of medication in the patient, in milligrams, is approaching 12.

6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 parts (c) and (d) on this page.

Response for question 6(c)

$$\frac{dy}{dt} = \frac{12-y}{3}$$

$$\int \frac{dy}{12-y} = \int \frac{dt}{3}$$

$$-\ln|12-y| = \frac{1}{3}t + C$$

$$-\ln|12-0| = \frac{1}{3}(0) + C$$

$$C = -\ln 12$$

$$-\ln|12-y| = \frac{1}{3}t - \ln 12$$

$$\ln|12-y| = -\frac{1}{3}t + \ln 12$$

$$e^{\ln|12-y|} = e^{-\frac{1}{3}t + \ln 12}$$

$$12-y = e^{-\frac{1}{3}t} \cdot e^{\ln 12}$$

$$12-y = 12e^{-\frac{1}{3}t}$$

$$-y = 12e^{-\frac{1}{3}t} - 12$$

$$\boxed{y = -12e^{-\frac{1}{3}t} + 12}$$

Response for question 6(d)

$$\cancel{\left.\frac{dy}{dt}\right|_{(t,y)=(1,2.5)} = 3 - \frac{2.5}{1+2} = 3 - \frac{2.5}{3} = 3 - \frac{5}{6} = +\#}$$

$$\frac{dy}{dt} = 3 - \frac{y}{t+2}$$

$$\frac{d^2y}{dt^2} = -\frac{dy/dx\,(t+2) - 1(y)}{(t+2)^2}$$

$$\frac{d^2y}{dt^2} = -\frac{\left(3 - \frac{y}{t+2}\right)(t+2) - y}{(t+2)^2}$$

$$\left.\frac{d^2y}{dt^2}\right|_{(1,2.5)} = -\frac{\left(3 - \frac{2.5}{1+2}\right)(1+2) - 2.5}{(1+2)^2} = -\frac{\left(3 - \frac{5}{6}\right)(3) - 2.5}{3^2} = -\#$$

The rate of change of the amount of medication in the second patient is decreasing at $t=1$ because $\left.\frac{d^2y}{dt^2}\right|_{(t,y)=(1,2.5)}$ is <0.

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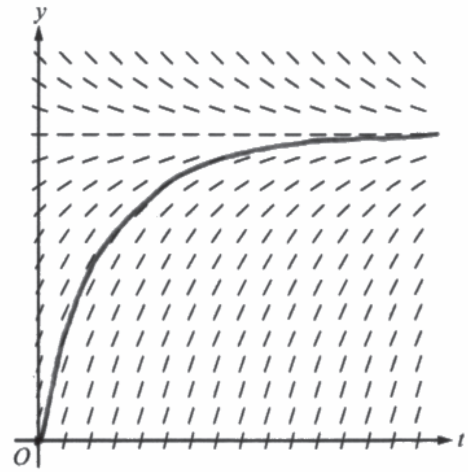
Answer QUESTION 6 parts (a) and (b) on this page.

Response for question 6(a)



Response for question 6(b)

$\lim_{t \to \infty} A(t) = 12$ means that 12 milligrams is the maximum amount of medication a patient can receive.

6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 parts (c) and (d) on this page.

Response for question 6(c)

$3dy = (12-y)\,dt$

$\int \frac{3}{12-y}\,dy = \int dt$

$-3\ln|12-y| = t + c$

$e^{\ln|12-y|} = e^{-\frac{1}{3}t + \frac{1}{3}c}$

$12 - y = e^{-\frac{1}{3}t - \frac{1}{3}c}$

$y = 12 - e^{-\frac{1}{3}t - \frac{1}{3}c}$

$0 = 12 - e^{-\frac{1}{3}(0) - \frac{1}{3}c}$

$0 = 12 - e^{-\frac{1}{3}c}$

$e^{-\frac{c}{3}} = 12$

$-\frac{c}{3} = \ln 12$

$c = -3\ln 12$

$\boxed{y = 12 - e^{-\frac{t}{3} + \ln 12}}$

Response for question 6(d)

$\frac{d^2y}{dt^2} = -\frac{(t+2) - y}{(t+2)^2}$

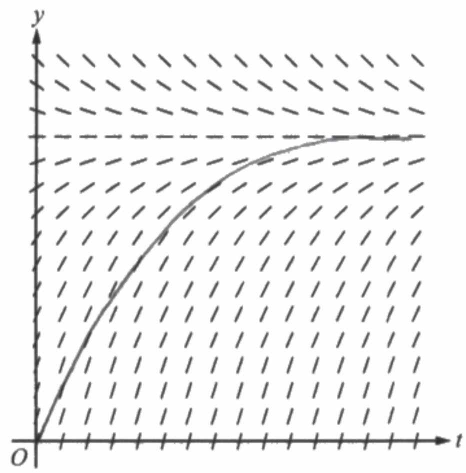
$\frac{d^2y}{dt^2} = -\frac{(1+2) - 2.5}{(1+2)^2} = -\frac{0.5}{9} = -\frac{1}{18}$

The rate of change of the amount of medication is decreasing because the second derivative is less than 0 at $t=1$.

6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 parts (a) and (b) on this page.

Response for question 6(a)



Response for question 6(b)

As t aproaches ∞ the amount of medication in the patient aproaches 12 milligrams of medication.

Answer QUESTION 6 parts (c) and (d) on this page.

Response for question 6(c)

$$\frac{dy}{dt} = \frac{12-y}{3}$$

$$\frac{dy}{dt} + \frac{y}{3} = 4$$

$$A(t) + \frac{y^2}{6} = 4y$$

$$\boxed{A(t) = 4y - \frac{y^2}{6}}$$

Response for question 6(d)

$$\frac{dy}{dt} = 3 - \frac{y}{t+2}$$

$$\frac{dy}{dt} \text{ at } t=1 = 3 - \frac{2.5}{3}$$

$$\frac{d^2y}{dt^2} = -\left(\frac{(t+2)\left(\frac{dy}{dx}\right) - (y)(1)}{(t+2)^2}\right)$$

$$= -\left(\frac{(3)\left(3 - \frac{2.5}{3}\right) - (2.5)}{9}\right)$$

The rate of change of the amount of medication in the second patient is decreasing, as $B''(t)$ is negative.

Question 6

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem a function $y = A(t)$ models the amount of medication, in milligrams, in a patient at time t hours. This function satisfies the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$, and at time $t = 0$ hours, there are 0 milligrams of medication in the patient.

In part (a) students were shown a portion of the slope field for the given differential equation and asked to sketch the solution curve through the point $(0, 0)$. A correct response would draw a single increasing, concave down curve starting at $(0, 0)$, approaching the horizontal asymptote with slopes equal to zero from below.

In part (b) students were asked to interpret the statement $\lim_{x \to \infty} A(t) = 12$ using correct units in this context. A correct response would indicate this statement means that over time the amount of medication in the patient approaches 12 milligrams.

In part (c) students were asked to use separation of variables to find the particular solution $y = A(t)$ with $A(0) = 0$. A correct response should separate the variables, integrate, and use the initial condition $A(0) = 0$ to resolve the constant of integration and arrive at the solution $A(t) = 12 - 12e^{-t/3}$.

In part (d) a second function $y = B(t)$, which satisfies $\frac{dy}{dt} = 3 - \frac{y}{t + 2}$, is introduced as a model for the amount of medication in a second patient at time t hours. At time $t = 1$ hour, there are 2.5 milligrams of medication in the second patient. Students were asked whether the amount of medication in the patient is increasing or decreasing at time $t = 1$. A correct response would use the quotient rule to compute $B''(t) = \frac{d^2y}{dt^2}$, determine that $B'(1) \neq 0$ and $B''(1) < 0$, and then would conclude the amount of medication is decreasing.

Sample: 6A
Score: 9

The response earned 9 points: 1 point in part (a), 1 point in part (b), 4 points in part (c), and 3 points in part (d). In part (a) the response earned the point because the solution curve passes through $(0, 0)$, is increasing and concave down, and approaches the horizontal asymptote. In part (b) the response was earned because it references both "medication in the patient in milligrams" and "approaching 12." In part (c) the response earned the first point on line 2 on the left side for a correct separation of variables. The second point was earned on line 3 on the left side for correct antiderivatives. The third point was earned for the "$+ c$" on line 3 on the left side with the use of the initial condition on line 4 on the left side. The fourth point was earned for a correct solution presented in the box on the last line on the right side. In part (d) the response earned the first point on line 2 for the correct second derivative expression. It is unclear on lines 3 and the first part of line 4 whether or not the leading negative sign is in the numerator or in front of the fraction; however, it is clear on the final presented numerical value. The second point was earned for the correct numeric expression of the second derivative. The expression does not have to be simplified to earn the point. The third point was earned for the answer decreasing with the reasoning based on the sign of the second derivative at $(1, 2.5)$.

Question 6 (continued)

Sample: 6B
Score: 6

The response earned 6 points: 1 point in part (a), no points in part (b), 4 points in part (c), and 1 point in part (d). In part (a) the response earned the point because the solution curve passes through $(0, 0)$, is increasing and concave down, and approaches the horizontal asymptote. In part (b) the point was not earned because the response states "12 milligrams is the maximum amount" instead of "approaches 12 milligrams" and the response refers to the amount of medication "a patient can receive" instead of the amount of medication in the patient. In part (c) the response earned the first point on line 2 on the left side for the correct separation of variables. The second point was earned on line 3 on the left side for the correct antiderivatives. The third point was earned for the "$+ c$" on line 3 on the right side with the use of the initial condition on line 2 on the right side. The fourth point was earned for a correct form of the solution presented in the box on the last line. In part (d) the first point was not earned because the expression for the second derivative is incorrect. There is a missing $\frac{dy}{dt}$ in the numerator. The response is not eligible for the second point because the second point cannot be earned without the correct second derivative. The third point was earned for the conclusion, decreasing, based on the reasoning "because the second derivative is less than 0 at $t = 1$."

Sample: 6C
Score: 4

The response earned 4 points: 1 point in part (a), 1 point in part (b), no points in part (c), and 2 points in part (d). In part (a) the response earned the point because the solution curve passes through $(0, 0)$, is increasing and concave down, and approaches the horizontal asymptote. In part (b) the response earned the point because it mentions "amount of medication in the patient" and "aproaches 12 milligrams." In part (c) the response did not earn the first point because it does not separate the variables. A response that does not separate the variables is not eligible for any other points in part (c). In part (d) the response earned the first point on line 2 for the correct second derivative expression. The second point was not earned because equating the symbolic second derivative to the numeric second derivative expression causes a linkage error. In this case, if the linkage error had not occurred, the point would be earned for the correct unsimplified value of the second derivative at $t = 1$. The third point was earned for the answer decreasing with the reason "$B''(t)$ is negative."