2022

AP® CollegeBoard

AP® Calculus AB

Sample Student Responses and Scoring Commentary

Inside:

Free-Response Question 2

☑ Scoring Guidelines

☑ Student Samples

☑ Scoring Commentary

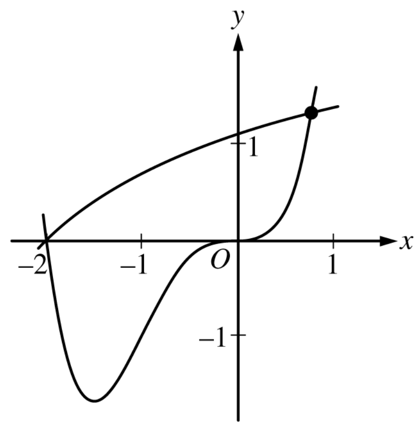
Part A (AB): Graphing calculator required

Question 2 — 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.



Let f and g be the functions defined by $f(x) = \ln(x+3)$ and $g(x) = x^4 + 2x^3$. The graphs of f and g, shown in the figure above, intersect at $x = -2$ and $x = B$, where $B > 0$.

	Model Solution	Scoring	
(a)	Find the area of the region enclosed by the graphs of f and g.		
	$\ln(x+3) = x^4 + 2x^3 \Rightarrow x = -2,\ x = B = 0.781975$	Integrand	**1 point**
	$\int_{-2}^{B} (f(x) - g(x))\,dx = 3.603548$	Limits of integration	**1 point**
	The area of the region is 3.604 (or 3.603).	Answer	**1 point**

Scoring notes:

- Other forms of the integrand in a definite integral, e.g., $|f(x) - g(x)|$, $|g(x) - f(x)|$, or $g(x) - f(x)$, earn the first point.
- To earn the second point, the response must have a lower limit of -2 and an upper limit expressed as either the letter B with no value attached, or a number that is correct to the number of digits presented, with at least one and up to three decimal places.
    - Case 1: If the response did not earn the second point because of an incorrect value of B, $0 < B < 1$, but used a lower limit of -2, the response earns the third point only for a consistent answer.
    - Case 2: If the response did not earn the second point because the lower limit used was $x = 0$, but the response used a correct upper limit of B, the response earns the third point for a consistent answer of 0.708 (or 0.707).
    - Case 3: If a response uses any other incorrect limits it does not earn the second or third points.
- A response containing the integrand $g(x) - f(x)$ must interpret the value of the resulting integral correctly to earn the third point. For example, the following response earns all 3 points: $\int_{-2}^{B}(g(x) - f(x))\,dx = -3.604$ so the area is 3.604. However, the response "Area $= \int_{-2}^{B}(g(x) - f(x))\,dx = 3.604$" presents an untrue statement and earns the first and second points but not the third point.
- A response must earn the first point in order to be eligible for the third point. If the response has earned the second point, then only the correct answer will earn the third point.
- Instructions for scoring a response that presents an integrand of $\ln(x + 3) - x^4 + 2x^3$ and the correct answer are shown in the "Global Special Case" after part (d).

Total for part (a) 3 points

(b) For $-2 \le x \le B$, let $h(x)$ be the vertical distance between the graphs of f and g. Is h increasing or decreasing at $x = -0.5$? Give a reason for your answer.

$h(x) = f(x) - g(x)$ $h'(x) = f'(x) - g'(x)$ $h'(-0.5) = f'(-0.5) - g'(-0.5) = -0.6$ (or -0.599)	Considers $h'(-0.5)$ – OR – $f'(x) - g'(x)$	**1 point**
Since $h'(-0.5) < 0$, h is decreasing at $x = -0.5$.	Answer with reason	**1 point**

Scoring notes:

- The response need not present the value of $h'(-0.5)$. The last line earns both points. However, if a value is presented it must be correct for the digits reported up to three decimal places.
- A response that reports an incorrect value of $h'(-0.5)$ earns only the first point.
- A response that presents only $h'(x)$ does not earn either point.
- The only response that earns the second point for concluding "h is increasing" is described in the "Global Special Case" provided after part (d).
- A response that compares the values of $f'(x)$ and $g'(x)$ at $x = -0.5$ earns the first point and is eligible for the second point. This comparison can be made symbolically or verbally; for example, the response "the rate of change of $f(x)$ is less than the rate of change of $g(x)$ at $x = -0.5$" earns the first point.

Total for part (b) 2 points

(c) The region enclosed by the graphs of f and g is the base of a solid. Cross sections of the solid taken perpendicular to the x-axis are squares. Find the volume of the solid.

$\int_{-2}^{B}(f(x)-g(x))^2\,dx = 5.340102$	Integrand **1 point**
The volume of the solid is 5.340.	Answer **1 point**

Scoring notes:

- The first point is earned for an integrand of $k(f(x)-g(x))^2$ or its equivalent with $k \neq 0$ in any definite integral. If $k \neq 1$, then the response is not eligible for the second point.
- A response that does not earn the first point is ineligible to earn the second point, with the following exceptions:
  - A response which has a presentation error in the integrand (for example, mismatched or missing parentheses, misplaced exponent) does not earn the first point but would earn the second point for the correct answer. A response which has a presentation error in the integrand and which reports an incorrect answer earns no points.
  - A response that presents an integrand of $\left(\ln(x+3)-x^4+2x^3\right)^2$. Scoring instructions for this case are provided in the “Global Special Case” after part (d).
- A response that uses incorrect limits is only eligible for the second point, provided the limits are imported from part (a) in Case 1 or Case 2. In both of these situations, the second point is earned only for answers consistent with the imported limits.

Total for part (c) 2 points

(d) A vertical line in the xy-plane travels from left to right along the base of the solid described in part (c). The vertical line is moving at a constant rate of 7 units per second. Find the rate of change of the area of the cross section above the vertical line with respect to time when the vertical line is at position $x = -0.5$.

The cross section has area $A(x) = (f(x)-g(x))^2$. $\frac{d}{dt}[A(x)] = \frac{dA}{dx}\cdot\frac{dx}{dt}$	$\frac{dA}{dx}\cdot\frac{dx}{dt}$ **1 point**
$\frac{d}{dt}[A(x)]\Big\rvert_{x=-0.5} = A'(-0.5)\cdot 7 = -9.271842$ At $x=-0.5$, the area of the cross section above the line is changing at a rate of -9.272 (or -9.271) square units per second.	Answer **1 point**

Scoring notes:

- The first point may be earned by presenting $\frac{dA}{dx} \cdot \frac{dx}{dt}$, $A' \cdot \frac{dx}{dt}$, $A'(x) \cdot \frac{dx}{dt}$, $A'(x) \cdot x'$, $A'(-0.5) \cdot 7$, or $k \cdot 7$, where k is a declared value of $A'(-0.5)$, or any equivalent expression, including $2(f(x) - g(x))(f'(x) - g'(x))\frac{dx}{dt}$.
- If a response defines $f(x) - g(x)$ as a function in parts (b) or (c) (for example, $h(x) = f(x) - g(x)$), then a correct expression for $\frac{dA}{dt}$ (for example, $2h\frac{dh}{dt}$) earns the first point.
- A response that imports a function $A(x)$ declared in part (c) is eligible for both points (the answer must be consistent with the imported function $A(x)$).
- A response that presents an incorrect function for $A(x)$ that is not imported from part (c) is eligible only for the first point.
- Except when $A(x)$ is imported from part (c), the second point is earned only for the correct answer.
- A response that does not earn the first point is ineligible to earn the second point except in the special case noted below.

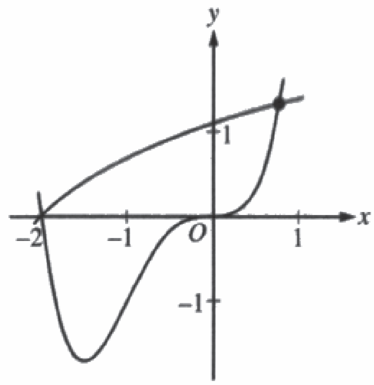
Total for part (d) 2 points

Total for question 2 9 points

Global Special Case: A response may incorrectly simplify $f(x) - g(x)$ to $j(x) = \ln(x + 3) - x^4 + 2x^3$ instead of $\ln(x + 3) - x^4 - 2x^3$. Because this question is calculator active, a response with this incorrect simplification may nevertheless present correct answers.

- In any part of the question, a response that starts correctly by using $f(x) - g(x)$, then presents $j(x)$, is eligible for all points in that part.
- The first time a response implicitly presents $f(x) - g(x)$ as $j(x) = \ln(x + 3) - x^4 + 2x^3$ (with no explicit connection) in any part of this question, the response loses a point. The response is then eligible for all remaining points for a correct or consistent answer.
- In part (a) the consistent answer using $j(x)$ is negative and will not earn the third point.
- In part (b) the consistent answer using $j(x)$ is that $j'(-0.5) = 2.4 > 0$, so h is increasing at $x = -0.5$.
- In part (c) the consistent answer using $j(x)$ is 252.187 (or 252.188).
- In part (d) the consistent answer using $j(x)$ is 20.287.

Answer QUESTION 2 parts (a) and (b) on this page.



Response for question 2(a)

$B = .7819751$

$$\int_{-2}^{.7819751} \left(\ln(x+3) - (x^4 + 2x^3)\right) dx = 3.604$$

Response for question 2(b)

$h(x) = \ln(x+3) - (x^4 + 2x^3)$

$h(x)$ is decreasing at $x = -.5$ because $h'(x) < 0$

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Answer QUESTION 2 parts (c) and (d) on this page.

Response for question 2(c)

$A = s^2 \quad s = h(x)$

$V = \int_{-2}^{B} (h(x))^2 \, dx = 5.340$

Response for question 2(d)

$x = -.5$

$A = (h(x))^2$

$\frac{dx}{dt} = 7 \text{ u/s}$

$\frac{dA}{dt} = 2h(x) \cdot h'(x) \cdot \frac{dx}{dt}$

$\frac{dA}{dt} = 2(1.10379) \cdot (-.6) \cdot 7 = -9.272 \text{ units}^2/\text{second}$

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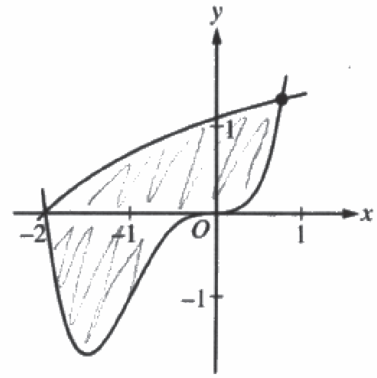
Answer QUESTION 2 parts (a) and (b) on this page.



Response for question 2(a)

B = 0.78198

$$\int_{-2}^{0.78198} (f(x) - g(x))\,dx = 3.6035$$

Response for question 2(b)

$$\int_{-2}^{-0.5} (f(x) - g(x))\,dx = 2.3657$$

$$\int_{-0.5}^{B} (f(x) - g(x))\,dx = 1.2378$$

Decreasing because the area between f(x) and g(x) is greater from ~~x=-2 to~~ x=-0.5 than from x=-0.5 to x=B.

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Q5108/6

Answer QUESTION 2 parts (c) and (d) on this page.

Response for question 2(c)

$$\int_{-2}^{B} (f(x)-g(x))^2 \, dx$$

$\downarrow$

$\boxed{5.3401}$

Response for question 2(d)

$A = (f(x) - g(x))$

$\frac{dA}{dt} = (f' \frac{dx}{dt} - g' \frac{dx}{dt})$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

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Q5108/7

Sample 2C

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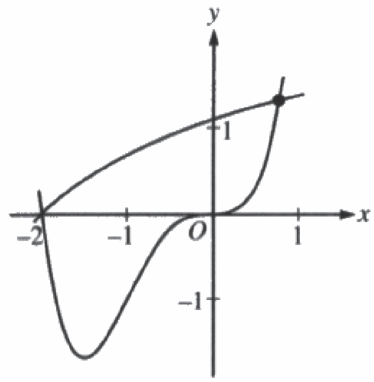
Answer QUESTION 2 parts (a) and (b) on this page.



Response for question 2(a)

$\int_{-2}^{.782} [f(x) - g(x)]\,dx = 12.022$

Response for question 2(b)

increasing at $t = -0.5$ b/c $y = .916$ and is both positive and above the x axis, so therefore its increasing on $t = -0.5$

Answer QUESTION 2 parts (c) and (d) on this page.

Response for question 2(c)

$\pi \int_{-2}^{.782} [f(x) - g(x)]^2 \, dx = 792.272 \text{ units}^3$

Response for question 2(d)

$At/_{dt} = 1.104$ units per second

$A'(t) = f(x) - g(x)$

$A'(-.5) = f(-.5) - g(-.5)$

Question 2

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem students were provided graphs of the functions $f(x) = \ln(x + 3)$ and $g(x) = x^4 + 2x^3$ and told that the graphs intersect at $x = -2$ and $x = B$, where $B > 0$.

In part (a) students were asked to find the area of the region enclosed by the graphs of f and g. A correct response provides the setup of the definite integral of $f(x) - g(x)$ from $x = -2$ to $x = B$. The response must determine the value of B (although this value need not be presented) and then use this value to evaluate the integral and find an area of 3.604.

In part (b) the function $h(x)$ is defined to be the vertical distance between the graphs of f and g, and students were asked to reason whether h is increasing or decreasing at $x = -0.5$. A correct response would recognize that the vertical distance between the graphs of f and g is $f(x) - g(x)$ and then evaluate the derivative $h'(x) = f'(x) - g'(x)$ at $x = -0.5$. Because this value is negative, the response should conclude that h is decreasing when $x = -0.5$.

In part (c) students were told that the region enclosed by the graphs of f and g is the base of a solid with cross sections of the solid taken perpendicular to the x-axis that are squares. Students were asked to find the volume of the solid. A correct response would realize that the area of a cross section is $(f(x) - g(x))^2$ and would find the requested volume by integrating this area from $x = -2$ to $x = B$.

In part (d) students were told that a vertical line in the xy-plane travels from left to right along the base of the solid described in part (c) at a constant rate of 7 units per second. Students were asked to find the rate of change of the area of the cross section above the vertical line with respect to time when the vertical line is at position $x = -0.5$. A correct response would again use the area function from part (c), $A(x) = (f(x) - g(x))^2$ and the chain rule to find $\frac{d}{dt}[A(x)] = \frac{dA}{dx} \cdot \frac{dx}{dt}$. The response should then use a calculator to find $A'(-0.5)$ and multiply this value by the given value of $\frac{dx}{dt} = 7$.

Sample: 2A
Score: 9

The response earned 9 points: 3 points in part (a), 2 points in part (b), 2 points in part (c), and 2 points in part (d).

In part (a) the response earned the first point with the correct integrand in a definite integral. The response earned the second point with correct limits on the definite integral. The response earned the third point with the circled correct answer.

In part (b) the response earned the first point by noting that $x = -0.5$ in the second line and stating that $h'(x) < 0$ in the third line. The response earned the second point with a correct answer in the second line and by stating that $h'(x) < 0$ at $x = -0.5$ in the second and third lines.

Question 2 (continued)

In part (c) the response earned the first point with the correct integrand in the definite integral. The function $h(x)$ is defined in part (b). The response is eligible for the second point because the limits of integration are -2 and B, for B defined in part (a). The response earned the second point with the circled correct answer.

In part (d) the response earned the first point with the expression $2h(x) \cdot h'(x) \cdot \frac{dx}{dt}$ in the second line on the left, which is equivalent to $\frac{dA}{dx} \cdot \frac{dx}{dt}$. The function $h(x)$ is defined in part (b). The response would have earned the second point with "$2(1.10379) \cdot (-.6) \cdot 7$" in the middle of the third line on the left but went on to simplify and earned the point with the correct circled answer on the same line. The response provides correct units, but these are not required to earn the second point.

Sample: 2B
Score: 6

The response earned 6 points: 3 points in part (a), no points in part (b), 2 points in part (c), and 1 point in part (d).

In part (a) the response earned the first point with the correct integrand in a definite integral. The response earned the second point with correct limits on the definite integral. The response earned the third point with the correct circled answer.

In part (b) the response did not earn the first point because it considers neither $f'(x) - g'(x)$ nor $h'(-0.5)$. Because the response did not earn the first point, it is not eligible for the second point.

In part (c) the response earned the first point with the correct integrand in the definite integral. The response earned the second point with the correct answer boxed in line 2.

In part (d) the response is eligible for the first point because the incorrect expression for A is a nonconstant function. The response earned the first point in the second line by expressing $\frac{dA}{dt}$ in a form equivalent to $\frac{dA}{dx} \cdot \frac{dx}{dt}$. The response is not eligible for the second point because the expression for A is incorrect.

Sample: 2C
Score: 3

The response earned 3 points: 2 points in part (a), no points in part (b), 1 point in part (c), and no points in part (d).

In part (a) the response earned the first point with a correct integrand in a definite integral. The response earned the second point with correct limits on the definite integral. The response did not earn the third point because the answer is incorrect.

In part (b) the response did not earn the first point because it considers neither $f'(x) - g'(x)$ nor $h'(-0.5)$. Because the response did not earn the first point, it is not eligible for the second point.

In part (c) the response earned the first point with an integrand of the form $k(f(x) - g(x))^2$ with $k \neq 0$ in a definite integral. Because $k \neq 1$, the response is not eligible for the second point.

Question 2 (continued)

In part (d) the response did not earn the first point because the expression for $\frac{dA}{dt}$ in the second line is not of the form $\frac{dA}{dx} \cdot \frac{dx}{dt}$. Because the response did not earn the first point, it is not eligible for the second point.