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AP[®]

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AP[®] Calculus AB

Sample Student Responses and Scoring Commentary

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Free-Response Question 4

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Part B (AB or BC): Graphing calculator not allowed**Question 4****9 points****General Scoring Notes**

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	-6.1	-5.0	-4.4	-3.8	-3.5

An ice sculpture melts in such a way that it can be modeled as a cone that maintains a conical shape as it decreases in size. The radius of the base of the cone is given by a twice-differentiable function r , where $r(t)$ is measured in centimeters and t is measured in days. The table above gives selected values of $r'(t)$, the rate of change of the radius, over the time interval $0 \leq t \leq 12$.

Model Solution**Scoring**

- (a) Approximate $r''(8.5)$ using the average rate of change of r' over the interval $7 \leq t \leq 10$. Show the computations that lead to your answer, and indicate units of measure.

$r''(8.5) \approx \frac{r'(10) - r'(7)}{10 - 7} = \frac{-3.8 - (-4.4)}{10 - 7}$	$r''(8.5)$ with supporting work	1 point
$= \frac{0.6}{3} = 0.2$ centimeter per day per day	Units	1 point

Scoring notes:

- To earn the first point the supporting work must include at least a difference and a quotient.
- Simplification is not required to earn the first point. If the numerical value is simplified, it must be correct.
- The second point can be earned with an incorrect approximation for $r''(8.5)$ but cannot be earned without some value for $r''(8.5)$ presented.
- Units may be written in any equivalent form (such as cm/day^2).

Total for part (a) 2 points

- (b) Is there a time t , $0 \leq t \leq 3$, for which $r'(t) = -6$? Justify your answer.

$r(t)$ is twice-differentiable. $\Rightarrow r'(t)$ is differentiable. $\Rightarrow r'(t)$ is continuous.	$r'(0) < -6 < r'(3)$	1 point
$r'(0) = -6.1 < -6 < -5.0 = r'(3)$ Therefore, by the Intermediate Value Theorem, there is a time t , $0 < t < 3$, such that $r'(t) = -6$.	Conclusion using Intermediate Value Theorem	1 point

Scoring notes:

- To earn the first point, the response must establish that -6 is between $r'(0)$ and $r'(3)$ (or -6.1 and -5). This statement may be represented symbolically (with or without including one or both endpoints in an inequality) or verbally. A response of “ $r'(t) = -6$ because $r'(0) = -6.1$ and $r'(3) = -5$ ” does not state that -6 is between -6.1 and -5 . Thus this response does not earn the first point.
- To earn the second point:
 - The response must state that $r'(t)$ is continuous because $r'(t)$ is differentiable (or because $r(t)$ is twice differentiable).
 - The response must have earned the first point.
 - Exception: A response of “ $r'(t) = -6$ because $r'(0) = -6.1$ and $r'(3) = -5$ ” does not earn the first point because of imprecise communication but may nonetheless earn the second point if all other criteria for the second point are met.
 - The response must conclude that there is a time t such that $r'(t) = -6$. (A statement of “yes” would be sufficient.)
- To earn the second point, the response need not explicitly name the Intermediate Value Theorem, but if a theorem is named, it must be correct.

Total for part (b) 2 points

- (c) Use a right Riemann sum with the four subintervals indicated in the table to approximate the value of $\int_0^{12} r'(t) dt$.

$\int_0^{12} r'(t) dt \approx 3r'(3) + 4r'(7) + 3r'(10) + 2r'(12)$	Form of right Riemann sum	1 point
$= 3(-5.0) + 4(-4.4) + 3(-3.8) + 2(-3.5)$ $= -51$	Answer	1 point

Scoring notes:

- To earn the first point, at least seven of the eight factors in the Riemann sum must be correct. If there is any error in the Riemann sum, the response does not earn the second point.
- A response of $3(-5.0) + 4(-4.4) + 3(-3.8) + 2(-3.5)$ earns both the first and second points, unless there is a subsequent error in simplification, in which case the response would earn only the first point.
- A response that presents the correct answer, with accompanying work that shows the four products in the Riemann sum (without explicitly showing all of the factors and/or the sum process) does not earn the first point but earns the second point. For example, $-15 + 4(-4.4) + 3(-3.8) + -7$ does not earn the first point but earns the second point. Similarly, $-15, -17.6, -11.4, -7 \rightarrow -51$ does not earn the first point but earns the second point.
- A response that presents the correct answer (-51) with no supporting work earns no points.
- A response that provides a completely correct left Riemann sum and approximation $\int_0^{12} r'(t) dt$ (i.e., $3r'(0) + 4r'(3) + 3r'(7) + 2r'(10) = 3(-6.1) + 4(-5.0) + 3(-4.4) + 2(-3.8) = -59.1$) earns 1 of the 2 points. A response that has any error in a left Riemann sum or evaluation for $\int_0^{12} r'(t) dt$ earns no points.
- Units are not required or read in this part.

Total for part (c) 2 points

- (d) The height of the cone decreases at a rate of 2 centimeters per day. At time $t = 3$ days, the radius is 100 centimeters and the height is 50 centimeters. Find the rate of change of the volume of the cone with respect to time, in cubic centimeters per day, at time $t = 3$ days. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

$\frac{dV}{dt} = \frac{2}{3}\pi r h \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt}$	Product rule	1 point
	Chain rule	1 point
$\frac{dV}{dt} \Big _{t=3} = \frac{2}{3}\pi(100)(50)(-5) + \frac{1}{3}\pi(100)^2(-2) = -\frac{70,000\pi}{3}$	Answer	1 point
The rate of change of the volume of the sculpture at $t = 3$ is approximately $-\frac{70,000\pi}{3}$ cubic centimeters per day.		

Scoring notes:

- The first 2 points could be earned in either order.
- A response with a completely correct product rule, missing one or both of the correct differentials, earns the product rule point, but not the chain rule point. For example, $\frac{dV}{dt} = \frac{2}{3}\pi r h + \frac{1}{3}\pi r^2$ earns the first point, but not the second.
- A response that treats r or h (but not both) as a constant is eligible for the chain rule point but not the product rule point. For example, $\frac{dV}{dt} = \frac{2}{3}\pi r h \frac{dr}{dt}$ is correct if h is constant, and thus earns the chain rule point.
- Note: Neither $\frac{dV}{dt} = \frac{2}{3}\pi r \frac{dh}{dt}$ nor $\frac{dV}{dt} = \frac{2}{3}\pi r h \frac{dr}{dt} \frac{dh}{dt}$ earns any points.
- A response that assumes a functional relationship between r and h (such as $r = 2h$), and uses this relationship to create a function for volume in terms of one variable, is eligible for at most the chain rule point. For example, $r = 2h \rightarrow V = \frac{1}{3}\pi(2h)^2 h = \frac{4}{3}\pi h^3 \rightarrow \frac{dV}{dt} = 4\pi h^2 \frac{dh}{dt}$ earns only the chain rule point.
- A response that mishandles the constant $\frac{1}{3}\pi$ cannot earn the third point but is eligible for the first 2 points.
- The third point cannot be earned without both of the first 2 points.
- $\frac{dV}{dt} = \frac{2}{3}\pi(100)(50)(-5) + \frac{1}{3}\pi(100)^2(-2)$ earns all 3 points.
- Units are not required or read in this part.

Total for part (d) 3 points**Total for question 4 9 points**

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NO CALCULATOR ALLOWED

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Answer QUESTION 4 parts (a) and (b) on this page.

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	-6.1	-5.0	-4.4	-3.8	-3.5

Response for question 4(a)

$$r''(8.5) \approx \frac{r'(10) - r'(7)}{10 - 7} = \frac{-3.8 - (-4.4)}{3} = \frac{-3.8 + 4.4}{3} = \frac{0.6}{3}$$

$$= 0.2 \text{ cm/day}^2$$

Response for question 4(b)

Since r is twice-differentiable, r' is also differentiable on the same interval $0 \leq t \leq 12$. Therefore r' is differentiable between $0 \leq t \leq 3$ as well since that interval is inside the previous one.

If a function is differentiable, it is also continuous. Therefore, $r'(t)$ is also continuous on $0 \leq t \leq 3$. According to the Intermediate Value Theorem, any continuous function $f(x)$ on (a,b) will take on every value between $f(a)$ and $f(b)$. Therefore $r'(t)$ will take on every value between -5.0 and -6.1 . -6 is inside this range, so there is a time t where $r'(t) = -6$ on $0 \leq t \leq 3$.

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Answer QUESTION 4 parts (c) and (d) on this page.

Response for question 4(c)

$$\begin{aligned}
 &(3-0)(-5.0) + (7-3)(-4.4) + (10-7)(-3.8) + (12-10)(-3.5) \\
 &3(-5) + 4(-4.4) + 3(-3.8) + 2(-3.5) \\
 &-15 + -17.6 - 11.4 - 7 \\
 &-15 - 29 - 7 \\
 &-44 - 7 \\
 &\boxed{-51}
 \end{aligned}$$

Response for question 4(d)

$$\frac{dh}{dt} = -2 \text{ cm/day}$$

$$r(3) = 100 \text{ cm}$$

$$h(3) = 50 \text{ cm}$$

$$r'(3) = \frac{dr}{dt} = -5.0 \text{ cm/day}$$

$$V = \frac{1}{3} \pi r^2 h$$

$$\frac{dV}{dt} = \frac{1}{3} \pi \left[(r^2) \left[\frac{dh}{dt} \right] + (h) \left[2r \frac{dr}{dt} \right] \right]$$

$$= \frac{1}{3} \pi \left[(100^2)(-2) + (50)(2(100)(-5.0)) \right]$$

$$= \frac{1}{3} \pi \left[-20000 + 50(-1000) \right]$$

$$= \frac{1}{3} \pi \left[-20000 + -50000 \right]$$

$$= \boxed{\frac{-70000\pi}{3} \text{ cm}^3/\text{day}}$$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

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Answer QUESTION 4 parts (a) and (b) on this page.

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	-6.1	-5.0	-4.4	-3.8	-3.5

Response for question 4(a)

$$\begin{array}{r} 4.4 \\ 3.8 \\ \hline 0.6 \end{array}$$

$$\frac{-3.8 - -4.4}{10 - 7} = \frac{0.6}{3} = \frac{\frac{6}{10}}{\frac{3}{1}} = \frac{6}{10} \cdot \frac{1}{3} = \frac{6}{30} = \frac{1}{5} \text{ centimeters/day}^2$$

Response for question 4(b)

$$\begin{array}{r} -5.0 + -6.1 \\ \hline 3 - 0 \end{array} = \frac{1.1}{3} = \frac{\frac{11}{10}}{\frac{3}{1}} = \frac{11}{30}$$

NO, the Mean Value theorem does not guarantee $r'(t) = -6$ on the interval $0 \leq t \leq 3$

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NO CALCULATOR ALLOWED

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Answer QUESTION 4 parts (c) and (d) on this page.

Response for question 4(c)

$$3(-5.0) + 4(-4.4) + 3(-3.8) + 2(-3.5)$$

$$-15 - 17.6 - 11.4 + 7 = -37.0$$

Response for question 4(d)

$$\frac{dh}{dt} = -2 \text{ cm/day} \quad t = 3 \text{ days}$$

$$r = 100 \text{ cm}$$

$$h = 50 \text{ cm}$$

$$V = \frac{1}{3} \pi r^2 h$$

$$\frac{dV}{dt} = \frac{2}{3} \pi r \frac{dr}{dt} \cdot h + \frac{dh}{dt} \cdot \frac{1}{3} \pi r^2$$

$$\frac{2}{3} \pi (100)$$

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NO CALCULATOR ALLOWED

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Answer QUESTION 4 parts (a) and (b) on this page.

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	-6.1	-5.0	-4.4	-3.8	-3.5

Response for question 4(a)

$$\frac{-3.8 - (-4.4)}{10 - 7} = \frac{0.6}{3} = \boxed{0.2}$$

Response for question 4(b)

Yes, because the Mean Value Theorem states that if a function is continuous and differentiable, there will be a c where $a \leq c \leq b$. Since -6.0 is between -6.1 and -5.0 in the interval $0 \leq t \leq 3$, there will be a value for r where $r'(r) = 6$.

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Answer QUESTION 4 parts (c) and (d) on this page.

Response for question 4(c)

$$\int_0^{12} r'(t) dt = 2(-3.5) + 3(-3.8) + 4(-4.4) + 3(-5.0)$$

$$= -7.0 - 11.4 - 17.6 - 15.0 = \boxed{-52.0}$$

Response for question 4(d)

$$\frac{dh}{dt} = -2$$

$$V = \frac{1}{3} \pi r^2 h$$

$$r = 100 \text{ cm}$$

$$h = 50 \text{ cm}$$

$$\frac{dV}{dt} = \frac{2}{3} \pi r \frac{dr}{dt} h + \frac{dV}{dt} = \frac{2}{3} \pi r h \frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{2}{3} \pi (100)(50)(-2)$$

$$\frac{dV}{dt} = \boxed{\frac{-20000}{3} \pi \frac{\text{cm}^3}{\text{day}}}$$

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Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Question 4

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem the melting of an ice sculpture can be modeled as a cone that maintains a conical shape as it decreases in size. The radius of the base of the cone is a twice-differentiable function $r(t)$ measured in centimeters, with time t , $0 \leq t \leq 12$, in days. Selected values of $r'(t)$ are provided in a table.

In part (a) students were asked to approximate $r''(8.5)$ using the average rate of change of r' over the interval $7 \leq t \leq 10$ and to provide correct units. A correct response should estimate the value using a difference quotient, drawing from the data in the table that most tightly bounds $t = 8.5$. The response should include units of centimeters per day per day.

In part (b) students were asked to justify whether there is a time t , $0 \leq t \leq 3$, for which the rate of change of r is equal to -6 . A correct response will use the Intermediate Value Theorem, first noting that the conditions for applying this theorem are met—specifically that $r'(t)$ is continuous because $r(t)$ is twice-differentiable and that -6 is bounded between the values of $r'(0)$ and $r'(3)$ given in the table. Therefore, by the Intermediate Value Theorem, there is a time t such that $0 < t < 3$, with $r'(t) = -6$.

In part (c) students were asked to use a right Riemann sum and the subintervals indicated by the table to approximate the value of $\int_0^{12} r'(t) dt$. A correct response should present the sum of the four products $\Delta t_i \cdot r'(t_i)$ drawn from the table and obtain an approximation value of -51 .

In part (d) students were told that the height of the cone decreases at a rate of 2 centimeters per day and that at time $t = 3$ the radius of the cone is 100 cm and the height is 50 cm. They are asked to find the rate of change of the volume of the cone with respect to time at time $t = 3$ days. A correct response will use the product and chain rules to differentiate the given function for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$, and then evaluate the resulting derivative using values $r = 100$, $h = 50$, $\left. \frac{dh}{dt} \right|_{t=3} = -2$, and $\left. \frac{dr}{dt} \right|_{t=3} = -5$ (from the table) to obtain a rate of $-\frac{70,000\pi}{3}$ cubic centimeters per day.

Sample: 4A

Score: 9

The response earned 9 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and 3 points in part (d).

In part (a) the response would have earned the first point with the expression $\frac{-3.8 - (-4.4)}{3}$ in line 1, with no simplification. In this case, correct simplification to the boxed answer of 0.2 in line 2 earned the first point. The response earned the second point for the correct units presented in the boxed answer in line 2.

Question 4 (continued)

In part (b) the response earned the first point with the statements given in the last three lines, which place the value -6 between the values of -5.0 and -6.1 . The response earned the second point with the statements in lines 1 through 5, concluding that $r'(t)$ is continuous because $r'(t)$ is differentiable. The response names the Intermediate Value Theorem, which is not required but is correct.

In part (c) the response earned the first point with the sum of products expression given in line 1. The response would have also earned the second point for this expression in line 1, with no further simplification. In this case, correct simplification to the boxed answer of -51 in line 6 earned the second point.

In part (d) the response earned the first and second points with the correct $\frac{dV}{dt}$ expression given in line 2. The response would have also earned the third point for the correct evaluation of this expression given in line 3, with no further simplification. In this case, correct simplification presented in the boxed answer in line 6 earned the third point.

Sample: 4B**Score: 5**

The response earned 5 points: 2 points in part (a), no points in part (b), 1 point in part (c), and 2 points in part (d).

In part (a) the response would have earned the first point with the expression $\frac{-3.8 - (-4.4)}{10 - 7}$ at the beginning of line 1, with no simplification. In this case, correct simplification to the boxed answer of $\frac{1}{5}$ at the end of line 1 earned the first point. The response earned the second point for the correct units presented in the boxed answer in line 1.

In part (b) the response does not establish that -6 is between $r'(0)$ and $r'(3)$; thus, it did not earn the first point. The response did not earn the second point because the conclusion of “NO” is incorrect.

In part (c) the response earned the first point with the sum of products given in line 1. The response would have also earned the second point for this expression in line 1, with no further simplification. In this case, the response did not earn the second point because of an incorrect simplification to -37.0 .

In part (d) the response earned the first and second points with the correct expression for $\frac{dV}{dt}$ given on line 3. The response did not earn the third point because this expression was never evaluated.

Sample: 4C**Score: 3**

The response earned 3 points: 1 point in part (a), 1 point in part (b), 1 point in part (c), and no points in part (d).

In part (a) the response would have earned the first point with the expression $\frac{-3.8 - (-4.4)}{10 - 7}$ in line 1, with no simplification. In this case, correct simplification to the final answer of 0.2 at the end of line 1 earned the first point. The response does not present correct units, so it did not earn the second point.

Question 4 (continued)

In part (b) the response earned the first point for the statement given in line 4: "... since -6 is between -6.1 and -5.0 ." The response does not establish that the continuity condition of the Intermediate Value Theorem has been met and incorrectly names the theorem as the Mean Value Theorem, so it did not earn the second point.

In part (c) the response earned the first point with the sum of products given in line 1. The response would have also earned the second point for this expression in line 1, with no further simplification. In this case, incorrect simplification leads to a final answer of -52 , so the response did not earn the second point.

In part (d) the response earned no points. The response does not present either a correct product rule or correct chain rule; thus, it did not earn either of the first two points and is not eligible for the third point.