2022

AP® CollegeBoard

AP® Calculus AB

Sample Student Responses and Scoring Commentary

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Free-Response Question 5

- ☑ Scoring Guidelines
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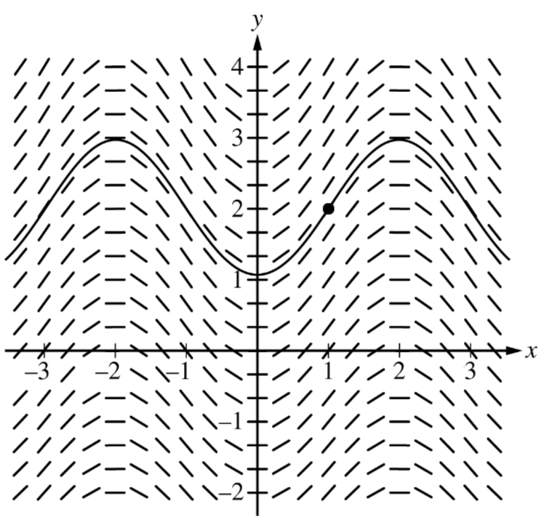
Part B (AB): Graphing calculator not allowed
Question 5 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

	Model Solution	Scoring	
(a)	A portion of the slope field for the differential equation is given below. Sketch the solution curve through the point $(1, 2)$.		
		Solution curve	**1 point**

Scoring notes:

- The solution curve must pass through the point $(1, 2)$, extend reasonably close to the left and right edges of the square and have no obvious conflicts with the given slope lines.
- Only portions of the solution curve within the given slope field are considered.
- The solution curve must indicate $f(x) > 0$ for all points on the curve.
- All local maximum/minimum points on the solution curve must occur at horizontal line segments in the slope field.

Total for part (a) **1 point**

(b) Write an equation for the line tangent to the solution curve in part (a) at the point $(1, 2)$. Use the equation to approximate $f(0.8)$.

$\left.\frac{dy}{dx}\right\|_{(x,\, y)=(1,\, 2)} = \frac{1}{2} \cdot 3 \cdot \sin\left(\frac{\pi}{2}\right) = \frac{3}{2}$ An equation for the tangent line is $y = 2 + \frac{3}{2}(x - 1)$.	Tangent line equation	**1 point**
$f(0.8) \approx 2 + \frac{3}{2}(0.8 - 1) = 1.7$	Approximation	**1 point**

Scoring notes:

- The tangent line equation can be presented in any equivalent form.
- An incorrect tangent line equation with a slope of $\frac{3}{2}$ is eligible to earn the second point for a consistent answer.
- A response of only $2 + \frac{3}{2}(0.8 - 1)$ earns the second point but not the first.

Total for part (b) 2 points

(c) It is known that $f''(x) > 0$ for $-1 \le x \le 1$. Is the approximation found in part (b) an overestimate or an underestimate for $f(0.8)$? Give a reason for your answer.

Because $f''(x) > 0$, f is concave up on $-1 \le x \le 1$, the tangent line lies below the graph of $y = f(x)$ at $x = 0.8$, and the approximation for $f(0.8)$ is an underestimate.	Answer with reason	**1 point**

Scoring notes:

- The reason must include $f''(x) > 0$, $f'(x)$ is increasing, or $f(x)$ is concave up.

Total for part (c) 1 point

(d) Use separation of variables to find $y = f(x)$, the particular solution to the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$ with the initial condition $f(1) = 2$.

$\int \frac{dy}{\sqrt{y+7}} = \int \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)dx$	Separation of variables	**1 point**
$2\sqrt{y+7} = -\frac{1}{\pi}\cos\left(\frac{\pi}{2}x\right) + C$	One correct antiderivative	**1 point**
	The other correct antiderivative	**1 point**
$f(1) = 2 \Rightarrow 2\sqrt{2+7} = -\frac{1}{\pi}\cos\left(\frac{\pi}{2}\cdot 1\right) + C$ $\Rightarrow 6 = -\frac{1}{\pi}\cos\left(\frac{\pi}{2}\right) + C \Rightarrow C = 6$ $\sqrt{y+7} = 3 - \frac{1}{2\pi}\cos\left(\frac{\pi}{2}x\right)$	Constant of integration and uses initial condition	**1 point**
$y = \left(3 - \frac{1}{2\pi}\cos\left(\frac{\pi}{2}x\right)\right)^2 - 7$	Solves for y	**1 point**

Scoring notes:

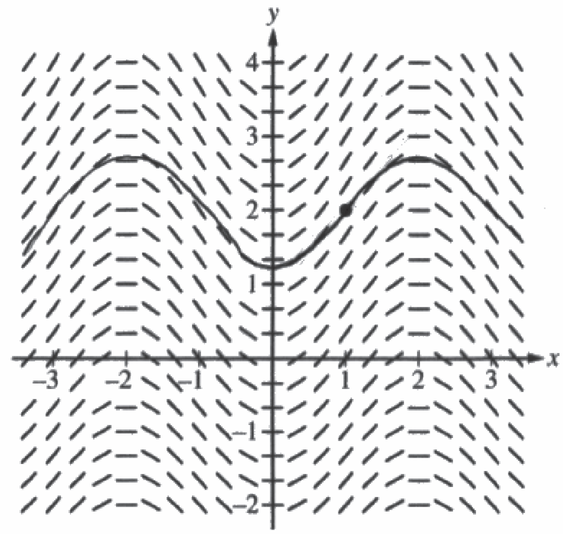
- A response with no separation of variables earns 0 out of 5 points.
- A response with no constant of integration can earn at most the first 3 points.
- A response is eligible for the fourth point only if it has earned the first point and at least 1 of the 2 antiderivative points.
  - Special case: The incorrect separation of $\sqrt{y+7}\ dy = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)dx$ does not earn the first point, is only eligible for the antiderivative point for $-\frac{1}{\pi}\cos\left(\frac{\pi}{2}x\right)$, and is eligible for the fourth point.
- An eligible response earns the fourth point by correctly including the constant of integration in an equation and substituting 1 for x and 2 for y.
- A response is eligible for the fifth point only if it has earned the first 4 points.

Total for part (d) **5 points**

Total for question 5 **9 points**

Answer QUESTION 5 parts (a) and (b) on this page.

Response for question 5(a)



Response for question 5(b)

$\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$

At $(1, 2)$:

$= \frac{1}{2}\sin\left(\frac{\pi}{2}\right)\sqrt{9}$

$= \frac{1}{2}\cdot 3 = \frac{3}{2}$

$y - y_1 = m(x - x_1)$

$y - 2 = \frac{3}{2}(x-1)$

$\boxed{y = \frac{3}{2}(x-1)+2}$

$y = \frac{3}{2}(.8-1)+2$

$\boxed{f(.8) = \frac{3}{2}(-.2)+2}$

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (c) and (d) on this page.

Response for question 5(c)

The approximation found in part(b) is an underestimate because the graph of f' is concave upward meaning that the tangent line would go under the curve making it an underestimation of the actual value.

Response for question 5(d)

$$\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$$

$$\int \frac{dy}{\sqrt{y+7}} = \int \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)dx$$

$$\int (y+7)^{-\frac{1}{2}}\,dy = \int \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)dx$$

$$2\sqrt{y+7} = \frac{1}{2}\cdot -\cos\left(\frac{\pi}{2}x\right)\cdot\frac{2}{\pi} + C$$

$$2\sqrt{y+7} = -\frac{1}{\pi}\cos\left(\frac{\pi}{2}x\right) + C$$

$$2\sqrt{9} = \frac{-1}{\pi}\cos\left(\frac{\pi}{2}\right) + C$$

$$6 = C$$

$$2\sqrt{y+7} = \frac{-1}{\pi}\cos\left(\frac{\pi}{2}x\right) + 6$$

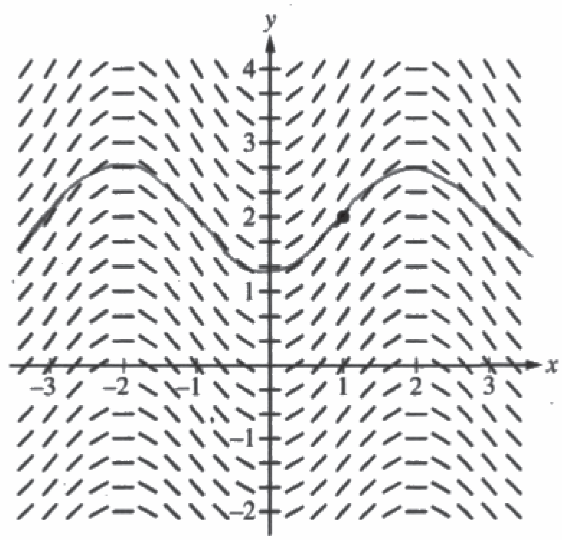
$$\boxed{y = \left(\frac{\frac{-1}{\pi}\cos\left(\frac{\pi}{2}x\right) + 6}{2}\right)^2 - 7}$$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (a) and (b) on this page.

Response for question 5(a)



Response for question 5(b)

$y - 2 = \frac{3}{2}(x - 1)$

$y - 2 = \frac{3}{2}(0.8 - 1)$ 1.50×0.20

0.2

$= 0.25$

$= 2.20$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

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Answer QUESTION 5 parts (c) and (d) on this page.

Response for question 5(c)

Approximation is underestimate because $f''(x) > 0$.

Response for question 5(d)

$$\int \frac{1}{\sqrt{y+7}}\, dy = \int \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\, dx$$

$$(y+7)^{-\frac{1}{2}}$$

$$\frac{(y)^{-\frac{1}{2}+1}}{2} = -\frac{1}{2}\cos\left(\frac{\pi}{2}x\right)\cdot\frac{\pi}{2}$$

$$2\left(\frac{\sqrt{y}}{2}\right) = \left(-\frac{\pi}{4}\cos\left(\frac{\pi}{2}x\right)\right)\cdot 2$$

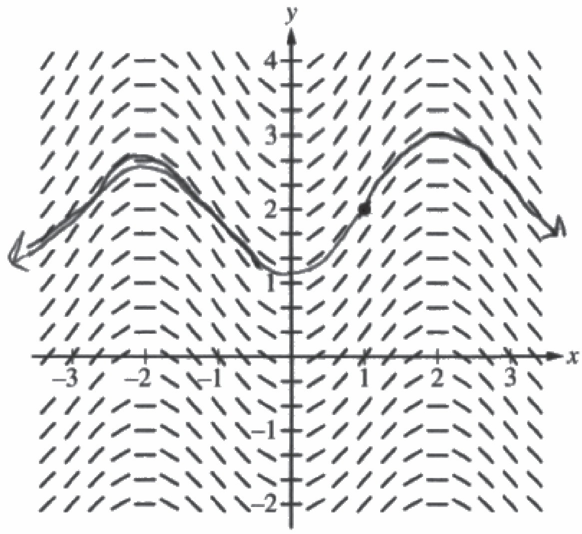
$$\sqrt{y}^2 = \left(-\frac{\pi}{4}\cos\left(\frac{\pi}{2}x\right)\right)^2\cdot 2^2$$

$$y = \left(-\frac{\pi}{4}\cos\left(\frac{\pi}{2}x\right)\right)^2\cdot 2^2$$

$$y = 2$$

Answer QUESTION 5 parts (a) and (b) on this page.

Response for question 5(a)



Response for question 5(b)

$y-2=m(x-1)$

$m=\frac{1}{2}\sin(\frac{\pi}{2})\sqrt{9}$

$m=\frac{1}{2}(1)(3)$

$y-2=\frac{3}{2}(0.8-1)$

$m=\frac{3}{2}$

$y=\frac{3}{2}(-.2)+2$

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (c) and (d) on this page.

Response for question 5(c)

Because $f''(x) > 0$ and concave up,
$f(0.6)$ was an underestimate

Response for question 5(d)

$$\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$$

$$\left(\frac{1}{\frac{1}{2}\sin\left(\frac{\pi}{2}x\right)} \cdot \frac{dy}{dx}\right)^2 = \sqrt{y+7}^{\,2}$$

$$-7 + \frac{1}{\frac{1}{2}\sin\left(\frac{\pi}{2}x\right)} = y$$

Question 5

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem students were given a differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$ and told that $y = f(x)$ is the particular solution to the equation with initial condition $f(1) = 2$. They are also told that f is defined for all real numbers.

In part (a) a portion of the slope field for this differential equation is shown, and students were asked to sketch the solution curve through the point $(1, 2)$. A correct response will draw a curve that follows the indicated slope segments in the first and second quadrants, through the point $(1, 2)$, with minimum and maximum points occurring at horizontal line segments on the slope field.

In part (b) students were asked to write an equation for the line tangent to the solution curve in part (a) at the point $(1, 2)$ and to use that equation to approximate $f(0.8)$. A correct response would use the given differential equation to find the slope of the tangent line, $\left.\frac{dy}{dx}\right|_{(x,\, y)=(1,\, 2)} = \frac{3}{2}$, then use this slope and the given point to find a tangent line equation of $y = 2 + \frac{3}{2}(x-1)$. Additionally, the response should substitute $x = 0.8$ in the tangent line equation to obtain an approximation of $f(0.8) \approx 1.7$.

In part (c) students were told that $f''(x) > 0$ for $-1 \le x \le 1$ and asked to reason whether the approximation found in part (b) is an over- or underestimate for $f(0.8)$. A correct response will reason that $f''(x) > 0$ on $-1 \le x \le 1$ means f is concave up on $-1 \le x \le 1$; therefore, the tangent line lies below the graph of $y = f(x)$, and the approximation is an underestimate of $f(0.8)$.

In part (d) students were asked to use separation of variables to find the particular solution $y = f(x)$ to the given differential equation with initial condition $f(1) = 2$. A correct response should separate the variables, integrate, use the initial condition $f(1) = 2$ to determine the value of the constant of integration, and arrive at the solution of $y = \left(3 - \frac{1}{2\pi}\cos\left(\frac{\pi}{2}x\right)\right)^2 - 7$.

Sample: 5A
Score: 8

The response earned 8 points: 1 point in part (a), 2 points in part (b), no points in part (c), and 5 points in part (d).

In part (a) the response earned the point with a correct solution curve passing through the point $(1, 2)$.

Question 5 (continued)

In part (b) the response would have earned the first point in line 2 on the right side for the equation $y - 2 = \frac{3}{2}(x - 1)$ but continued to simplify and earned the first point for the boxed answer in line 3 on the right side with a correct equation for the tangent line. The response would have earned the second point for the unsimplified approximation in line 4 on the right side but continued to simplify and earned the second point for the boxed answer in line 5 on the right side with a correct approximation of $f(0.8)$.

In part (c) the response correctly determines the approximation is an underestimate but did not earn the point because of the incorrect statement in line 2, "f' is concave upward."

In part (d) the response earned the first point in line 2 for a correct separation of variables. The second point was earned in line 4 on the left side of the equation for the correct antiderivative. The third point was earned in line 5 on the right side of the equation for the correct antiderivative. This antiderivative is initiated in line 4 with an unclear use of the negative sign, but this is clarified in line 5. The fourth point was earned for the correct use of "$+C$" in line 4 and for using the initial condition in line 6. The fifth point was earned for the boxed answer with a correct expression for the particular solution.

Sample: 5B
Score: 4

The response earned 4 points: 1 point in part (a), 1 point in part (b), 1 point in part (c), and 1 point in part (d).

In part (a) the response earned the point with a correct solution curve passing through the point $(1, 2)$.

In part (b) the response earned the first point in line 1 with a correct equation for the tangent line. The response did not earn the second point because the approximation 2.25 is incorrect.

In part (c) the response earned the point by stating, "Approximation is underestimate because $f''(x) > 0$."

In part (d) the response earned the first point in line 1 for a correct separation of variables. The second and third points were not earned because of the incorrect antiderivatives $\frac{(y)^{-\frac{1}{2}+1}}{2}$ and $-\frac{1}{2}\cos\left(\frac{\pi}{2}x\right)\cdot\frac{\pi}{2}$ in line 3. Because neither antiderivative point was earned, the response is not eligible for the fourth or fifth points.

Sample: 5C
Score: 3

The response earned 3 points: 1 point in part (a), 2 points in part (b), no points in part (c), and no points in part (d).

In part (a) the response earned the point with a correct solution curve passing through the point $(1, 2)$. The curve does not have to be symmetric with respect to the y-axis to earn the point.

In part (b) the response earned the first point in line 1 on the left side with a correct equation for the tangent line and by clearly defining m in line 3 on the right side. The equation $y - 2 = m(x - 1)$ alone would not earn the first point. The response earned the second point for the circled approximation.

Question 5 (continued)

In part (c) the response correctly concludes " $f(0.8)$ was an underestimate" but did not earn the point because the reasoning "Because $f''(x) > 0$ and concave up" implies $f''(x)$ is concave up, which is incorrect.

In part (d) the response earned no points because there is no acceptable separation of variables.