2023

AP

AP® Calculus AB

Sample Student Responses and Scoring Commentary

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Free-Response Question 6

- ☑ Scoring Guidelines
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Part B (AB): Graphing calculator not allowed
Question 6 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Consider the curve given by the equation $6xy = 2 + y^3$.

Model Solution	Scoring	

(a) Show that $\frac{dy}{dx} = \frac{2y}{y^2 - 2x}$.

$\frac{d}{dx}(6xy) = \frac{d}{dx}\left(2 + y^3\right) \Rightarrow 6y + 6x\frac{dy}{dx} = 3y^2\frac{dy}{dx}$	Implicit differentiation	**1 point**
$\Rightarrow 2y = \frac{dy}{dx}\left(y^2 - 2x\right) \Rightarrow \frac{dy}{dx} = \frac{2y}{y^2 - 2x}$	Verification	**1 point**

Scoring notes:

- The first point is earned only for the correct implicit differentiation of $6xy = 2 + y^3$. Responses may use alternative notations for $\frac{dy}{dx}$, such as y'.
- The second point cannot be earned without the first point.
- It is sufficient to present $2y = \frac{dy}{dx}\left(y^2 - 2x\right)$ to earn the second point, provided there are no subsequent errors.

Total for part (a) 2 points

(b) Find the coordinates of a point on the curve at which the line tangent to the curve is horizontal, or explain why no such point exists.

For the line tangent to the curve to be horizontal, it is necessary that $2y = 0$ (so $y = 0$) and that $y^2 - 2x \neq 0$.	Sets $2y = 0$	**1 point**
Substituting $y = 0$ into $6xy = 2 + y^3$ yields the equation $6x \cdot 0 = 2$, which has no solution. Therefore, there is no point on the curve at which the line tangent to the curve is horizontal.	Answer with reason	**1 point**

Scoring notes:

- The first point is earned with any of $2y = 0$, $y = 0$, $\frac{dy}{dx} = 0$, $dy = 0$, $y' = 0$, or $\frac{2y}{y^2 - 2x} = 0$.
- A response need not state that at a horizontal tangent, $y^2 - 2x \neq 0$.

Total for part (b) 2 points

(c) Find the coordinates of a point on the curve at which the line tangent to the curve is vertical, or explain why no such point exists.

For a line tangent to this curve to be vertical, it is necessary that $2y \neq 0$ and that $y^2 - 2x = 0$ (so $x = \frac{y^2}{2}$).	Sets $y^2 - 2x = 0$	**1 point**
Substituting $x = \frac{y^2}{2}$ into $6xy = 2 + y^3$ yields the equation $3y^2 \cdot y = 2 + y^3 \Rightarrow 2y^3 = 2 \Rightarrow y = 1$.	Substitutes $x = \frac{y^2}{2}$ into $6xy = 2 + y^3$	**1 point**
Substituting $y = 1$ in $6xy = 2 + y^3$ yields $6x = 2 + 1$, or $x = \frac{1}{2}$. The tangent line to the curve is vertical at the point $\left(\frac{1}{2}, 1\right)$.	Answer	**1 point**

Scoring notes:

- The first point can be earned by presenting $y^2 = 2x$ or $y = \sqrt{2x}$.
- The second point can be earned for the substitution of $y = \sqrt{2x}$ into $6xy = 2 + y^3$, or for substituting $x = \frac{2 + y^3}{6y}$ into $y^2 - 2x = 0$.
- A response earns all three points by setting $y^2 - 2x = 0$, declaring the point $\left(\frac{1}{2}, 1\right)$, and verifying that this point is on the curve $6xy = 2 + y^3$.
- A response that identifies the point $\left(\frac{1}{2}, 1\right)$ but does not verify that the point is on the curve, does not earn the second or the third point.
- To earn the third point the response must present both coordinates of the point $\left(\frac{1}{2}, 1\right)$. The coordinates need not appear as an ordered pair as long as they are labeled.

Total for part (c) 3 points

(d) A particle is moving along the curve. At the instant when the particle is at the point $\left(\frac{1}{2}, -2\right)$, its horizontal position is increasing at a rate of $\frac{dx}{dt} = \frac{2}{3}$ unit per second. What is the value of $\frac{dy}{dt}$, the rate of change of the particle's vertical position, at that instant?

$6y\frac{dx}{dt} + 6x\frac{dy}{dt} = 0 + 3y^2\frac{dy}{dt}$	Uses implicit differentiation with respect to t	**1 point**
At the point $(x, y) = \left(\frac{1}{2}, -2\right)$, $6(-2)\left(\frac{2}{3}\right) + 6\left(\frac{1}{2}\right)\frac{dy}{dt} = 3(-2)^2\frac{dy}{dt}$ $\Rightarrow -8 + 3\frac{dy}{dt} = 12\frac{dy}{dt}$ $\Rightarrow \frac{dy}{dt} = -\frac{8}{9}$ unit per second	Answer	**1 point**

Scoring notes:

- The first point is earned by presenting one or more of the terms $6y\frac{dx}{dt}$, $6x\frac{dy}{dt}$, or $3y^2\frac{dy}{dt}$.
- Units will not affect scoring in this part.
- An unsupported response of $-\frac{8}{9}$ earns no points.
- Alternate solution:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\left.\frac{dy}{dx}\right|_{(x,\, y)=(1/2,\, -2)} = \frac{2(-2)}{(-2)^2 - 2(1/2)} = -\frac{4}{3}$$

$$\left.\frac{dy}{dt}\right|_{(x,\, y)=(1/2,\, -2)} = \left.\frac{dy}{dx} \cdot \frac{dx}{dt}\right|_{(x,\, y)=(1/2,\, -2)} = -\frac{4}{3} \cdot \frac{2}{3} = -\frac{8}{9} \text{ unit per second}$$

  - The first point is earned for the statement $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$ or equivalent.
  - A numerical expression, such as $-\frac{4}{3} \cdot \frac{2}{3}$ or $\frac{2(-2)}{(-2)^2 - 2\left(\frac{1}{2}\right)} \cdot \frac{2}{3}$, earns both points.

Total for part (d) 2 points

Total for question 6 9 points

6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 parts (a) and (b) on this page.

Response for question 6(a)

$6xy = 2 + y^3$

$6y + 6x\frac{dy}{dx} = 3y^2\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{6y}{3y^2 - 6x}$

$\boxed{\frac{dy}{dx} = \frac{2y}{y^2 - 2x}}$

Response for question 6(b)

$\frac{dy}{dx} = \frac{2y}{y^2 - 2x}$

horizontal tangent : $2y = 0$

$y = 0$

$6x(0) = 2 + 0$

$0 \neq 2$

therefore, since there are no x-values at y=0, there are no points with a horizontal tangent

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

0045048

Q5171/14

6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 parts (c) and (d) on this page.

Response for question 6(c)

$\frac{dy}{dx} = \frac{2y}{y^2-2x}$

vertical tangent: $y^2-2x=0$

$x = \frac{y^2}{2}$

$6(\frac{y^2}{2})(y) = 2+y^3$

$3y^3 = 2+y^3$

$2y^3 = 2$

$y^3 = 1 \rightarrow y = 1, \ x = \frac{1}{2}$

at $(\frac{1}{2}, 1)$, there is a vertical tangent

Response for question 6(d)

$6xy = 2+y^3$

$6y\frac{dx}{dt} + 6x\frac{dy}{dt} = 3y^2\frac{dy}{dt}$

$\frac{dy}{dt} = \frac{6y\, dx/dt}{3y^2-6x}$

$\frac{dy}{dt} = \frac{6(-2)(2/3)}{3(-2)^2-6(1/2)} = \frac{-8}{12-3}$

$\frac{dy}{dt} = -\frac{8}{9}$ units per second

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Q5171/15

6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 parts (a) and (b) on this page.

Response for question 6(a)

$y^3+2=6xy$

$3y^2\frac{dy}{dx}=6y+6x\frac{dy}{dx}$

$3y^2\frac{dy}{dx}-6x\frac{dy}{dx}=6y$

$\frac{dy}{dx}(3y^2-6x)=6y$

$\frac{dy}{dx}=\frac{6y}{3y^2-6x}=\frac{2y}{y^2-2x}$

Response for question 6(b)

$0=\frac{2y}{y^2-2x}$

$y=0$

$\frac{0}{0-2x}=\frac{0}{0-2x}$

$(0,0)$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

0130200

Q5171/14

6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 parts (c) and (d) on this page.

Response for question 6(c)

DNE since vertical tangent cannot exist on a function

Response for question 6(d)

$\left(\frac{1}{2}, -2\right)$

$2+y^3 = 6xy$

$2+3y^2\frac{dy}{dt} = 6\frac{dx}{dt}y + 6x\frac{dy}{dt}$

$2+12\frac{dy}{dt} = -6 + 3\frac{dy}{dt}$

$12\frac{dy}{dt} - 3\frac{dy}{dt} = 10$

$9\frac{dy}{dt} = 10$

$\frac{dy}{dt} = \frac{10}{9}$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 parts (a) and (b) on this page.

Response for question 6(a)

$6xy = 2 + y^3$

$6\left(x\frac{dy}{dx}\ y\frac{dx}{dy}\right) = 3y^2\frac{dy}{dx}$

$x\frac{dy}{dx}\ y\frac{dx}{dy} = \frac{1}{2}.y^2\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{2y}{y^2-2x}$

Response for question 6(b)

$(0,0)$

$2y = 0$

$y = 0$

$y^2 - 2x = 0$

$y^2 = 2x$

$y = \sqrt{2x}$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

0131666

Q5216/14

6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 parts (c) and (d) on this page.

Response for question 6(c)

$y^2 - 2x = 0$

$y^2 = 2x$

$y = \sqrt{2x}$

$(0, \sqrt{2x})$

Response for question 6(d)

$\frac{dy}{dt} = \frac{2y}{y^2 - 2x}$

$\frac{dy}{dt} = \frac{2(-2)}{(-2)^2 - 2(1/2)} = \frac{-4}{4-1} = \boxed{-\frac{4}{5}}$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Question 6

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

This problem asked students to consider the curve defined by the equation $6xy = 2 + y^3$.

In part (a) students were asked show that $\frac{dy}{dx} = \frac{2y}{y^2 - 2x}$. A correct response will implicitly differentiate the equation $6xy = 2 + y^3$ with respect to x, then solve the resulting equation for $\frac{dy}{dx}$.

In part (b) students were asked to find the coordinates of a point on the curve at which the tangent line is *horizontal*, or to explain why no such point exists. A correct response will note that a horizontal tangent line must have $\frac{dy}{dx} = 0$, which requires $2y = 0$ and, therefore, $y = 0$. But if $y = 0$, using the given equation $6xy = 2 + y^3$ yields $6x \cdot 0 = 2$, which has no solution. Therefore, there is no point on this curve at which the tangent line is horizontal.

In part (c) students were asked to find the coordinates of a point on the curve at which the tangent line is *vertical*, or to explain why no such point exists. A correct response will begin by noting that such a point requires $y^2 - 2x = 0 \Rightarrow x = \frac{y^2}{2}$. Substituting into the equation $6xy = 2 + y^3$ yields $y = 1$ and then $x = \frac{1}{2}$, resulting in a vertical tangent line at the point $\left(\frac{1}{2}, 1\right)$.

In part (d) students were asked to find the value of $\frac{dy}{dt}$ at the instant when the particle is at the point $\left(\frac{1}{2}, -2\right)$, given that at that instant the particle's horizontal position is increasing at a rate of $\frac{dx}{dt} = \frac{2}{3}$. A correct response will implicitly differentiate the equation $6xy = 2 + y^3$ with respect to t and then solve the resulting equation for $\frac{dy}{dt}$ using $x = \frac{1}{2}$, $y = -2$, and $\frac{dx}{dt} = \frac{2}{3}$.

Sample: 6A
Score: 9

The response earned 9 points: 2 points in part (a), 2 points in part (b), 3 points in part (c), and 2 points in part (d).

In part (a) the response earned the first point on the second line with the equation $6y + 6x\frac{dy}{dx} = 3y^2\frac{dy}{dx}$, the correct implicit differentiation for the given curve. The response correctly solves for $\frac{dy}{dx}$ on the third line and then earned the second point on the last line with the boxed equation.

Question 6 (continued)

In part (b) the response earned the first point on the second line with the equation $2y = 0$. The response earned the second point with the correct answer that "there are no points with a horizontal tangent," together with reason that "there are no x-values at $y = 0$."

In part (c) the response earned the first point on the second line with the equation $y^2 - 2x = 0$. The response then earned the second point on the fourth line with the correct substitution. The response would have earned the third point on the seventh line with the statements $y = 1$ and $x = \frac{1}{2}$; however, the response restates the answer correctly as an ordered pair and earned the point with the boxed section on the last line.

In part (d) the response earned the first point on the second line with the correct implicit differentiation of the curve with respect to t. The response would have earned the second point with the middle expression on the fourth line; however, the response presents two correct simplifications of this numerical answer and earned the point with the boxed answer.

Sample: 6B
Score: 4

The response earned 4 points: 2 points in part (a), 1 point in part (b), no points in part (c), and 1 point in part (d).

In part (a) the response earned the first point with the equation on the second line. The response correctly solves for $\frac{dy}{dx}$ on the subsequent three lines and earned the second point on the last line with the final simplification.

In part (b) the response earned the first point on the first line with the equation $0 = \frac{2y}{y^2 - 2x}$. The response then concludes that $y = 0$, which would also have earned the first point. The response did not earn the second point as there is no conclusion stating that no point exists.

In part (c) the response did not earn the first point as there is no evidence that the denominator of our presented $\frac{dy}{dx}$ has been set equal to 0. The response does not indicate any substitution, so it did not earn the second point. Finally, as the correct point is not presented, the response did not earn the third point.

In part (d) the response states that $2 + 3y^2 \frac{dy}{dy} = 6\frac{dx}{dt}y + 6x\frac{dy}{dt}$ on the third line. While this is not the correct implicit differentiation of the given curve with respect to t, because at least one of the three terms involving the rates $\frac{dx}{dt}$ or $\frac{dy}{dt}$ is correct, the response earned the first point on this line. The response did not earn the second point because the answer presented is not correct.

Sample: 6C
Score: 2

The response earned 2 points: no points in part (a), 1 point in part (b), 1 point in part (c), and no points in part (d).

In part (a) the response did not earn the first point as no correct implicit differentiation of the given curve is presented, except for the one that was given in the stem of the problem. Because the first point was not earned, the response did not earn the second point.

Question 6 (continued)

In part (b) the response earned the first point with the equation $2y = 0.$ Note that the next equation $y = 0$ would also have earned this point. As the response never concludes that no such point exists, the response did not earn the second point.

In part (c) the response earned the first point on the first line with the equation $y^2 - 2x = 0.$ Note that any of the first three lines would have earned this first point. As there is no substitution presented and the correct answer is not stated, the response earned neither the second point nor the third point.

In part (d) the response states on the first line that $\frac{dy}{dt} = \frac{2y}{y^2 - 2x}$ and then uses this expression as the basis for substitution with the given values for x and $y.$ If this expression is viewed as the implicit differentiation of the curve with respect to $t,$ then the response does not present at least one correct term with a rate $\frac{dx}{dt}$ or $\frac{dy}{dt}.$ If, on the other hand, this expression is meant to be $\frac{dy}{dx},$ then the solution presented never makes use of $\frac{dx}{dt}.$ In either case, the response did not earn the first point. As the stated answer is incorrect, the response did not earn the second point.