AP® Calculus BC
1999 Scoring Guidelines

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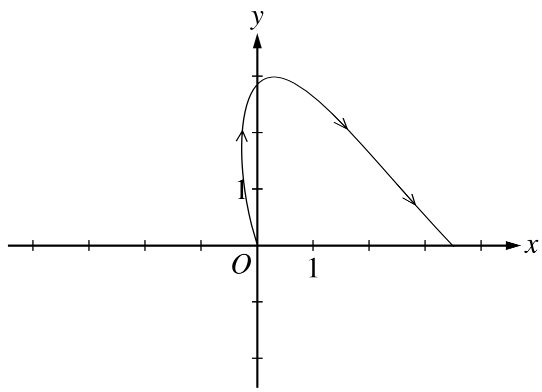
1. A particle moves in the xy–plane so that its position at any time t, $0 \le t \le \pi$, is given by $x(t) = \dfrac{t^2}{2} - \ln(1+t)$ and $y(t) = 3\sin t$.

   (a) Sketch the path of the particle in the xy–plane below. Indicate the direction of motion along the path.

   (b) At what time t, $0 \le t \le \pi$, does $x(t)$ attain its minimum value? What is the position $(x(t), y(t))$ of the particle at this time?

   (c) At what time t, $0 < t < \pi$, is the particle on the y–axis? Find the speed and the acceleration vector of the particle at this time.

(a)



$2 \begin{cases} 1\text{: graph} \\ 1\text{: direction} \end{cases}$

(b) $x'(t) = t - \dfrac{1}{1+t} = 0$

$t^2 + t - 1 = 0$

$t = \dfrac{-1+\sqrt{5}}{2}$ or $t = 0.618$ in $[0, \pi]$

$x(0.618) = -0.290 \qquad y(0.618) = 1.738$

$3 \begin{cases} 1\text{: } x'(t) = 0 \\ 1\text{: solution for } t \\ 1\text{: position} \end{cases}$

(c) $x(t) = \dfrac{t^2}{2} - \ln(1+t) = 0$

$t = 1.285$ or 1.286

$x'(t) = t - \dfrac{1}{1+t} \qquad y'(t) = 3\cos t$

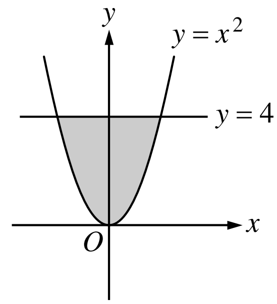
speed $= \sqrt{(x'(1.286))^2 + (y'(1.286))^2} = 1.196$

$x''(t) = 1 + \dfrac{1}{(1+t)^2} \qquad y''(t) = -3\sin t$

acceleration vector $= \langle x''(1.286), y''(1.286) \rangle$
$= \langle 1.191, -2.879 \rangle$

$4 \begin{cases} 1\text{: } x(t) = 0 \\ 1\text{: solution for } t \\ 1\text{: speed} \\ 1\text{: acceleration vector} \end{cases}$

2. The shaded region, R, is bounded by the graph of $y = x^2$ and the line $y = 4$, as shown in the figure above.
   (a) Find the area of R.
   (b) Find the volume of the solid generated by revolving R about the x–axis.
   (c) There exists a number k, $k > 4$, such that when R is revolved about the line $y = k$, the resulting solid has the same volume as the solid in part (b). Write, but do not solve, an equation involving an integral expression that can be used to find the value of k.



(a) Area $= \displaystyle\int_{-2}^{2} (4 - x^2)\, dx$

$$= 2\int_{0}^{2} (4 - x^2)\, dx$$

$$= 2\left[4x - \frac{x^3}{3}\right]_0^2$$

$$= \frac{32}{3} = 10.666 \text{ or } 10.667$$

2 { 1: integral
1: answer }

(b) Volume $= \displaystyle\pi\int_{-2}^{2} \left(4^2 - (x^2)^2\right) dx$

$$= 2\pi\int_{0}^{2} (16 - x^4)\, dx$$

$$= 2\pi\left[16x - \frac{x^5}{5}\right]_0^2$$

$$= \frac{256\pi}{5} = 160.849 \text{ or } 160.850$$

3 { 1: limits and constant
1: integrand
1: answer }

(c) $\displaystyle\pi\int_{-2}^{2} \left[(k - x^2)^2 - (k - 4)^2\right] dx = \frac{256\pi}{5}$

4 { 1: limits and constant
2: integrand
< −1 > each error
1: equation }

3. The rate at which water flows out of a pipe, in gallons per hour, is given by a differentiable function R of time t. The table above shows the rate as measured every 3 hours for a 24–hour period.

t (hours)	$R(t)$ (gallons per hour)
0	9.6
3	10.4
6	10.8
9	11.2
12	11.4
15	11.3
18	10.7
21	10.2
24	9.6

(a) Use a midpoint Riemann sum with 4 subdivisions of equal length to approximate $\int_0^{24} R(t)\,dt$. Using correct units, explain the meaning of your answer in terms of water flow.

(b) Is there some time t, $0 < t < 24$, such that $R'(t) = 0$? Justify your answer.

(c) The rate of water flow $R(t)$ can be approximated by $Q(t) = \frac{1}{79}\left(768 + 23t - t^2\right)$. Use $Q(t)$ to approximate the average rate of water flow during the 24–hour time period. Indicate units of measure.

(a) $\int_0^{24} R(t)\,dt \approx 6[R(3) + R(9) + R(15) + R(21)]$

$= 6[10.4 + 11.2 + 11.3 + 10.2]$

$= 258.6$ gallons

This is an approximation to the total flow in gallons of water from the pipe in the 24–hour period.

$$3\begin{cases} 1\text{: } R(3) + R(9) + R(15) + R(21) \\ 1\text{: answer} \\ 1\text{: explanation} \end{cases}$$

(b) Yes;

Since $R(0) = R(24) = 9.6$, the Mean Value Theorem guarantees that there is a t, $0 < t < 24$, such that $R'(t) = 0$.

$$2\begin{cases} 1\text{: answer} \\ 1\text{: MVT or equivalent} \end{cases}$$

(c) Average rate of flow

$\approx$ average value of $Q(t)$

$$= \frac{1}{24}\int_0^{24} \frac{1}{79}(768 + 23t - t^2)\,dt$$

$= 10.785$ gal/hr or 10.784 gal/hr

$$3\begin{cases} 1\text{: limits and average value constant} \\ 1\text{: } Q(t) \text{ as integrand} \\ 1\text{: answer} \end{cases}$$

(units) Gallons in part (a) and gallons/hr in part (c), or equivalent.

1: units

4. The function f has derivatives of all orders for all real numbers x. Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.
   (a) Write the third–degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1.5)$.
   (b) The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \le 3$ for all x in the closed interval $[1.5, 2]$. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \ne -5$.
   (c) Write the fourth–degree Taylor polynomial, $P(x)$, for $g(x) = f(x^2 + 2)$ about $x = 0$. Use P to explain why g must have a relative minimum at $x = 0$.

(a) $T_3(f,2)(x) = -3 + 5(x-2) + \frac{3}{2}(x-2)^2 - \frac{8}{6}(x-2)^3$

$$f(1.5) \approx T_3(f,2)(1.5)$$
$$= -3 + 5(-0.5) + \frac{3}{2}(-0.5)^2 - \frac{4}{3}(-0.5)^3$$
$$= -4.958\overline{3} = -4.958$$

4 $\begin{cases} 3\text{: } T_3(f,2)(x) \\ \quad <-1> \text{ each error} \\ 1\text{: approximation of } f(1.5) \end{cases}$

(b) Lagrange Error Bound $= \frac{3}{4!}|1.5 - 2|^4 = 0.0078125$

$f(1.5) > -4.958\overline{3} - 0.0078125 = -4.966 > -5$

Therefore, $f(1.5) \ne -5$.

2 $\begin{cases} 1\text{: value of Lagrange Error Bound} \\ 1\text{: explanation} \end{cases}$

(c) $P(x) = T_4(g,0)(x)$

$$= T_2(f,2)(x^2+2) = -3 + 5x^2 + \frac{3}{2}x^4$$

The coefficient of x in $P(x)$ is $g'(0)$. This coefficient is 0, so $g'(0) = 0$.

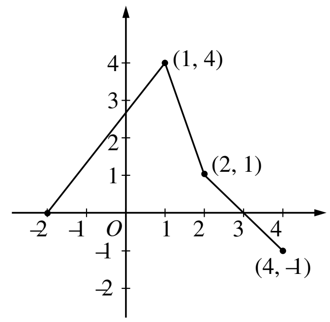
The coefficient of x^2 in $P(x)$ is $\frac{g''(0)}{2!}$. This coefficient is 5, so $g''(0) = 10$ which is greater than 0.

Therefore, g has a relative minimum at $x = 0$.

3 $\begin{cases} 2\text{: } T_4(g,0)(x) \\ \quad <-1> \text{ each incorrect, missing,} \\ \qquad \text{or extra term} \\ 1\text{: explanation} \end{cases}$

Note:
$<-1>$ max for improper use of $+\ldots$ or equality

5. The graph of the function f, consisting of three line segments, is given above. Let $g(x) = \int_1^x f(t)\,dt$.



   (a) Compute $g(4)$ and $g(-2)$.
   (b) Find the instantaneous rate of change of g, with respect to x, at $x = 1$.
   (c) Find the absolute minimum value of g on the closed interval $[-2, 4]$. Justify your answer.
   (d) The second derivative of g is not defined at $x = 1$ and $x = 2$. How many of these values are x–coordinates of points of inflection of the graph of g? Justify your answer.

(a) $g(4) = \int_1^4 f(t)\,dt = \frac{3}{2} + 1 + \frac{1}{2} - \frac{1}{2} = \frac{5}{2}$ $g(-2) = \int_1^{-2} f(t)\,dt = -\frac{1}{2}(12) = -6$	$2 \begin{cases} 1\text{: } g(4) \\ 1\text{: } g(-2) \end{cases}$
(b) $g'(1) = f(1) = 4$	**1**: answer
(c) g is increasing on $[-2, 3]$ and decreasing on $[3, 4]$. Therefore, g has absolute minimum at an endpoint of $[-2, 4]$. Since $g(-2) = -6$ and $g(4) = \frac{5}{2}$, the absolute minimum value is -6.	$3 \begin{cases} 1\text{: interior analysis} \\ 1\text{: endpoint analysis} \\ 1\text{: answer} \end{cases}$
(d) One; $x = 1$ On $(-2, 1)$, $g''(x) = f'(x) > 0$ On $(1, 2)$, $g''(x) = f'(x) < 0$ On $(2, 4)$, $g''(x) = f'(x) < 0$ Therefore $(1, g(1))$ is a point of inflection and $(2, g(2))$ is not.	$3 \begin{cases} 1\text{: choice of } x = 1 \text{ only} \\ 1\text{: show } (1, g(1)) \text{ is a point of inflection} \\ 1\text{: show } (2, g(2)) \text{ is not a point of inflection} \end{cases}$

6. Let f be the function whose graph goes through the point $(3, 6)$ and whose derivative is given by $f'(x) = \dfrac{1+e^x}{x^2}$.
   (a) Write an equation of the line tangent to the graph of f at $x = 3$ and use it to approximate $f(3.1)$.
   (b) Use Euler's method, starting at $x = 3$ with a step size of 0.05, to approximate $f(3.1)$. Use f'' to explain why this approximation is less than $f(3.1)$.
   (c) Use $\displaystyle\int_3^{3.1} f'(x)\,dx$ to evaluate $f(3.1)$.

(a) $f'(3) = \dfrac{1+e^3}{9} = 2.342$ or 2.343

$$y - 6 = \frac{1+e^3}{9}(x-3)$$

$$y = 6 + \frac{1+e^3}{9}(x-3)$$

$$f(3.1) \approx 6 + \frac{1+e^3}{9}(0.1) = 6.234$$

3 { 1: $f'(3)$; 1: equation; 1: approximation of $f(3.1)$ }

(b) $f(3.05) \approx f(3) + f'(3)(0.05)$

$= 6 + 0.11714 = 6.11714$

$f(3.1) \approx 6.11714 + f'(3.05)(0.05)$

$= 6.11714 + (2.37735)(0.05) = 6.236$

$$f''(x) = \frac{x^2e^x - 2x(1+e^x)}{x^4} = \frac{(x-2)e^x - 2}{x^3}$$

For $x \geq 3$, $f''(x) > \dfrac{e^x - 2}{x^3} > 0$ and the graph of f is concave upward on $(3, 3.1)$. Therefore, the Euler approximation lines at 3 and 3.05 lie below the graph.

4 { 1: Euler's method equations or equivalent table; 1: Euler approximation to $f(3.1)$ (not eligible without first point); 1: $f''(x)$; 1: reason }

(c) $f(3.1) - f(3) = \displaystyle\int_3^{3.1} \frac{1+e^x}{x^2}\,dx$

$f(3.1) = 6 + 0.2378 = 6.237$ or 6.238

2 { 1: $\displaystyle\int_3^{3.1} \frac{1+e^x}{x^2}\,dx = f(3.1) - f(3)$; 1: answer }