AP Calculus BC
2001 Scoring Guidelines

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Question 1

An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t with

$$\frac{dx}{dt} = \cos\left(t^3\right) \text{ and } \frac{dy}{dt} = 3\sin\left(t^2\right)$$

for $0 \le t \le 3$. At time $t = 2$, the object is at position (4,5).

(a) Write an equation for the line tangent to the curve at (4,5).

(b) Find the speed of the object at time $t = 2$.

(c) Find the total distance traveled by the object over the time interval $0 \le t \le 1$.

(d) Find the position of the object at time $t = 3$.

(a) $$\frac{dy}{dx} = \frac{3\sin\left(t^2\right)}{\cos\left(t^3\right)}$$

$$\left.\frac{dy}{dx}\right|_{t=2} = \frac{3\sin\left(2^2\right)}{\cos\left(2^3\right)} = 15.604$$

$$y - 5 = 15.604(x - 4)$$

1 : tangent line

(b) Speed $= \sqrt{\cos^2(8) + 9\sin^2(4)} = 2.275$

1 : answer

(c) Distance $= \int_0^1 \sqrt{\cos^2\left(t^3\right) + 9\sin^2\left(t^2\right)}\, dt$

$= 1.458$

$$3 : \begin{cases} 2 : \text{distance integral} \\ \quad <-1> \text{ each integrand error} \\ \quad <-1> \text{ error in limits} \\ 1 : \text{answer} \end{cases}$$

(d) $x(3) = 4 + \int_2^3 \cos\left(t^3\right) dt = 3.953$ or 3.954

$y(3) = 5 + \int_2^3 3\sin\left(t^2\right) dt = 4.906$

$$4 : \begin{cases} 1 : \text{definite integral for } x \\ 1 : \text{answer for } x(3) \\ 1 : \text{definite integral for } y \\ 1 : \text{answer for } y(3) \end{cases}$$

Question 2

The temperature, in degrees Celsius (°C), of the water in a pond is a differentiable function W of time t. The table above shows the water temperature as recorded every 3 days over a 15-day period.

t (days)	$W(t)$ (°C)
0	20
3	31
6	28
9	24
12	22
15	21

(a) Use data from the table to find an approximation for $W'(12)$. Show the computations that lead to your answer. Indicate units of measure.

(b) Approximate the average temperature, in degrees Celsius, of the water over the time interval $0 \le t \le 15$ days by using a trapezoidal approximation with subintervals of length $\Delta t = 3$ days.

(c) A student proposes the function P, given by $P(t) = 20 + 10te^{(-t/3)}$, as a model for the temperature of the water in the pond at time t, where t is measured in days and $P(t)$ is measured in degrees Celsius. Find $P'(12)$. Using appropriate units, explain the meaning of your answer in terms of water temperature.

(d) Use the function P defined in part (c) to find the average value, in degrees Celsius, of $P(t)$ over the time interval $0 \le t \le 15$ days.

(a) Difference quotient; e.g.

$$W'(12) \approx \frac{W(15) - W(12)}{15 - 12} = -\frac{1}{3} \text{ °C/day or}$$

$$W'(12) \approx \frac{W(12) - W(9)}{12 - 9} = -\frac{2}{3} \text{ °C/day or}$$

$$W'(12) \approx \frac{W(15) - W(9)}{15 - 9} = -\frac{1}{2} \text{ °C/day}$$

$$2 : \begin{cases} 1 : \text{difference quotient} \\ 1 : \text{answer (with units)} \end{cases}$$

(b) $\frac{3}{2}(20 + 2(31) + 2(28) + 2(24) + 2(22) + 21) = 376.5$

Average temperature $\approx \frac{1}{15}(376.5) = 25.1$ °C

$$2 : \begin{cases} 1 : \text{trapezoidal method} \\ 1 : \text{answer} \end{cases}$$

(c) $P'(12) = 10e^{-t/3} - \frac{10}{3}te^{-t/3}\Big|_{t=12}$

$= -30e^{-4} = -0.549$ °C/day

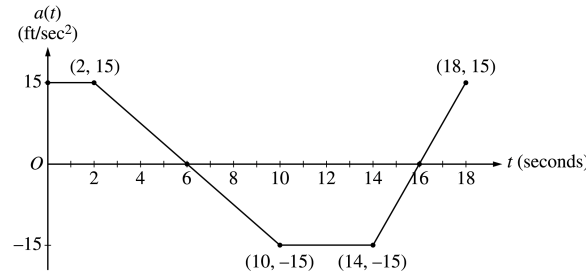
This means that the temperature is decreasing at the rate of 0.549 °C/day when $t = 12$ days.

$$2 : \begin{cases} 1 : P'(12) \text{ (with or without units)} \\ 1 : \text{interpretation} \end{cases}$$

(d) $\frac{1}{15}\int_0^{15}\left(20 + 10te^{-t/3}\right)dt = 25.757$ °C

$$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and} \\ \quad\ \text{average value constant} \\ 1 : \text{answer} \end{cases}$$

Question 3



A car is traveling on a straight road with velocity 55 ft/sec at time $t = 0$. For $0 \le t \le 18$ seconds, the car's acceleration $a(t)$, in ft/sec^2, is the piecewise linear function defined by the graph above.

(a) Is the velocity of the car increasing at $t = 2$ seconds? Why or why not?

(b) At what time in the interval $0 \le t \le 18$, other than $t = 0$, is the velocity of the car 55 ft/sec? Why?

(c) On the time interval $0 \le t \le 18$, what is the car's absolute maximum velocity, in ft/sec, and at what time does it occur? Justify your answer.

(d) At what times in the interval $0 \le t \le 18$, if any, is the car's velocity equal to zero? Justify your answer.

(a) Since $v'(2) = a(2)$ and $a(2) = 15 > 0$, the velocity is increasing at $t = 2$.

1 : answer and reason

(b) At time $t = 12$ because

$$v(12) - v(0) = \int_0^{12} a(t)\,dt = 0.$$

$$2 : \begin{cases} 1 : t = 12 \\ 1 : \text{reason} \end{cases}$$

(c) The absolute maximum velocity is 115 ft/sec at $t = 6$.

The absolute maximum must occur at $t = 6$ or at an endpoint.

$$v(6) = 55 + \int_0^6 a(t)\,dt$$
$$= 55 + 2(15) + \frac{1}{2}(4)(15) = 115 > v(0)$$

$$\int_6^{18} a(t)\,dt < 0 \text{ so } v(18) < v(6)$$

$$4 : \begin{cases} 1 : t = 6 \\ 1 : \text{absolute maximum velocity} \\ 1 : \text{identifies } t = 6 \text{ and} \\ \quad t = 18 \text{ as candidates} \\ \quad \text{or} \\ \quad \text{indicates that } v \text{ increases,} \\ \quad \text{decreases, then increases} \\ 1 : \text{eliminates } t = 18 \end{cases}$$

(d) The car's velocity is never equal to 0. The absolute minimum occurs at $t = 16$ where

$$v(16) = 115 + \int_6^{16} a(t)\,dt = 115 - 105 = 10 > 0.$$

$$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$$

Question 4

Let h be a function defined for all $x \neq 0$ such that $h(4) = -3$ and the derivative of h is given by $h'(x) = \dfrac{x^2 - 2}{x}$ for all $x \neq 0$.

(a) Find all values of x for which the graph of h has a horizontal tangent, and determine whether h has a local maximum, a local minimum, or neither at each of these values. Justify your answers.

(b) On what intervals, if any, is the graph of h concave up? Justify your answer.

(c) Write an equation for the line tangent to the graph of h at $x = 4$.

(d) Does the line tangent to the graph of h at $x = 4$ lie above or below the graph of h for $x > 4$? Why?

(a) $h'(x) = 0$ at $x = \pm\sqrt{2}$

$h'(x)$	$-$	0	$+$	und	$-$	0	$+$
x		$-\sqrt{2}$		0		$\sqrt{2}$	

Local minima at $x = -\sqrt{2}$ and at $x = \sqrt{2}$

$$4 : \begin{cases} 1 : x = \pm\sqrt{2} \\ 1 : \text{analysis} \\ 2 : \text{conclusions} \\ \quad < -1 > \text{ not dealing with discontinuity at } 0 \end{cases}$$

(b) $h''(x) = 1 + \dfrac{2}{x^2} > 0$ for all $x \neq 0$. Therefore, the graph of h is concave up for all $x \neq 0$.

$$3 : \begin{cases} 1 : h''(x) \\ 1 : h''(x) > 0 \\ 1 : \text{answer} \end{cases}$$

(c) $h'(4) = \dfrac{16 - 2}{4} = \dfrac{7}{2}$

$y + 3 = \dfrac{7}{2}(x - 4)$

1 : tangent line equation

(d) The tangent line is below the graph because the graph of h is concave up for $x > 4$.

1 : answer with reason

Question 5

Let f be the function satisfying $f'(x) = -3xf(x)$, for all real numbers x, with $f(1) = 4$ and $\lim_{x\to\infty} f(x) = 0$.

(a) Evaluate $\int_1^\infty -3xf(x)\,dx$. Show the work that leads to your answer.

(b) Use Euler's method, starting at $x = 1$ with a step size of 0.5, to approximate $f(2)$.

(c) Write an expression for $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = -3xy$ with the initial condition $f(1) = 4$.

(a) $\int_1^\infty -3xf(x)\,dx$

$$= \int_1^\infty f'(x)\,dx = \lim_{b\to\infty}\int_1^b f'(x)\,dx = \lim_{b\to\infty} f(x)\Big|_1^b$$

$$= \lim_{b\to\infty} f(b) - f(1) = 0 - 4 = -4$$

$2 : \begin{cases} 1 : \text{use of FTC} \\ 1 : \text{answer from limiting process} \end{cases}$

(b) $f(1.5) \approx f(1) + f'(1)(0.5)$

$$= 4 - 3(1)(4)(0.5) = -2$$

$f(2) \approx -2 + f'(1.5)(0.5)$

$$\approx -2 - 3(1.5)(-2)(0.5) = 2.5$$

$2 : \begin{cases} 1 : \text{Euler's method equations or equivalent table} \\ 1 : \text{Euler approximation to } f(2) \\ \quad\text{(not eligible without first point)} \end{cases}$

(c) $\frac{1}{y}\,dy = -3x\,dx$

$\ln y = -\frac{3}{2}x^2 + k$

$y = Ce^{-3/2\,x^2}$

$4 = Ce^{-3/2}$; $C = 4e^{3/2}$

$y = 4e^{3/2}\,e^{-3/2\,x^2}$

$5 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition } f(1) = 4 \\ 1 : \text{solves for } y \end{cases}$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

Question 6

A function f is defined by

$$f(x) = \frac{1}{3} + \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \cdots + \frac{n+1}{3^{n+1}}x^n + \cdots$$

for all x in the interval of convergence of the given power series.

(a) Find the interval of convergence for this power series. Show the work that leads to your answer.

(b) Find $\displaystyle\lim_{x\to 0} \frac{f(x) - \frac{1}{3}}{x}$.

(c) Write the first three nonzero terms and the general term for an infinite series that represents $\displaystyle\int_0^1 f(x)\,dx$.

(d) Find the sum of the series determined in part (c).

Solution	Scoring
(a) $\displaystyle\lim_{n\to\infty}\left\lvert \frac{\frac{(n+2)x^{n+1}}{3^{n+2}}}{\frac{(n+1)x^n}{3^{n+1}}} \right\rvert = \lim_{n\to\infty}\left\lvert \frac{(n+2)}{(n+1)}\frac{x}{3} \right\rvert = \left\lvert \frac{x}{3} \right\rvert < 1$ At $x = -3$, the series is $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{n+1}{3}$, which diverges. At $x = 3$, the series is $\displaystyle\sum_{n=0}^{\infty}\frac{n+1}{3}$, which diverges. Therefore, the interval of convergence is $-3 < x < 3$.	4 : { 1 : sets up ratio test 1 : computes limit 1 : conclusion of ratio test 1 : endpoint conclusion
(b) $\displaystyle\lim_{x\to 0}\frac{f(x)-\frac{1}{3}}{x} = \lim_{x\to 0}\left(\frac{2}{3^2} + \frac{3}{3^3}x + \frac{4}{3^4}x^2 + \cdots\right) = \frac{2}{9}$	1 : answer
(c) $\displaystyle\int_0^1 f(x)\,dx = \int_0^1\left(\frac{1}{3} + \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \cdots + \frac{n+1}{3^{n+1}}x^n + \cdots\right)dx$ $\displaystyle= \left(\frac{1}{3}x + \frac{1}{3^2}x^2 + \frac{1}{3^3}x^3 + \cdots + \frac{1}{3^{n+1}}x^{n+1} + \cdots\right)\Bigg\rvert_{x=0}^{x=1}$ $\displaystyle= \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{n+1}} + \cdots$	3 : { 1 : antidifferentiation of series 1 : first three terms for definite integral series 1 : general term
(d) The series representing $\displaystyle\int_0^1 f(x)\,dx$ is a geometric series. Therefore, $\displaystyle\int_0^1 f(x)\,dx = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$.	1 : answer