AP® Calculus BC
2002 Scoring Guidelines
Form B

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Question 1

A particle moves in the xy-plane so that its position at any time t, for $-\pi \le t \le \pi$, is given by $x(t) = \sin(3t)$ and $y(t) = 2t$.

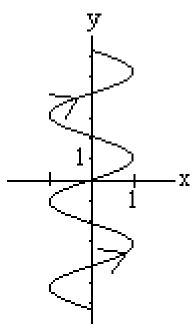
(a) Sketch the path of the particle in the xy-plane provided. Indicate the direction of motion along the path.

(b) Find the range of $x(t)$ and the range of $y(t)$.

(c) Find the smallest positive value of t for which the x-coordinate of the particle is a local maximum. What is the speed of the particle at this time?

(d) Is the distance traveled by the particle from $t = -\pi$ to $t = \pi$ greater than 5π? Justify your answer.

(a)



2 points:
- 1 : graph
  - three cycles of sine
  - x between -1 and 1
  - y between -2π and 2π
- 1 : direction

(b) $-1 \le x(t) \le 1$

$-2\pi \le y(t) \le 2\pi$

2 points:
- 1 : closed interval for $x(t)$
- 1 : closed interval for $y(t)$

(c) $x'(t) = 3\cos 3t = 0$

$3t = \dfrac{\pi}{2}$; $t = \dfrac{\pi}{6}$

Speed $= \sqrt{9\cos^2(3t) + 4}$

At $t = \dfrac{\pi}{6}$,

Speed $= \sqrt{9\cos^2\left(\dfrac{\pi}{2}\right) + 4} = 2$

3 points:
- 1 : $x'(t) = 3\cos 3t = 0$
- 1 : solves for t
- 1 : speed at student's time

(d) Distance $= \displaystyle\int_{-\pi}^{\pi} \sqrt{9\cos^2(3t) + 4}\, dt$

$= 17.973 > 5\pi$

2 points:
- 1 : integral for distance
- 1 : conclusion with justification

Question 2

The number of gallons, $P(t)$, of a pollutant in a lake changes at the rate $P'(t) = 1 - 3e^{-0.2\sqrt{t}}$ gallons per day, where t is measured in days. There are 50 gallons of the pollutant in the lake at time $t = 0$. The lake is considered to be safe when it contains 40 gallons or less of pollutant.

(a) Is the amount of pollutant increasing at time $t = 9$? Why or why not?

(b) For what value of t will the number of gallons of pollutant be at its minimum? Justify your answer.

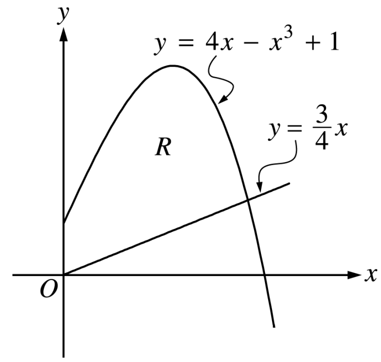
(c) Is the lake safe when the number of gallons of pollutant is at its minimum? Justify your answer.

(d) An investigator uses the tangent line approximation to $P(t)$ at $t = 0$ as a model for the amount of pollutant in the lake. At what time t does this model predict that the lake becomes safe?

(a) $P'(9) = 1 - 3e^{-0.6} = -0.646 < 0$ so the amount is not increasing at this time.	1 : answer with reason
(b) $P'(t) = 1 - 3e^{-0.2\sqrt{t}} = 0$ $t = (5\ln 3)^2 = 30.174$ $P'(t)$ is negative for $0 < t < (5\ln 3)^2$ and positive for $t > (5\ln 3)^2$. Therefore there is a minimum at $t = (5\ln 3)^2$.	3 { 1 : sets $P'(t) = 0$ 1 : solves for t 1 : justification
(c) $P(30.174) = 50 + \int_0^{30.174} \left(1 - 3e^{-0.2\sqrt{t}}\right) dt$ $= 35.104 < 40$, so the lake is safe.	3 { 1 : integrand 1 : limits 1 : conclusion with reason based on integral of $P'(t)$
(d) $P'(0) = 1 - 3 = -2$. The lake will become safe when the amount decreases by 10. A linear model predicts this will happen when $t = 5$.	2 { 1 : slope of tangent line 1 : answer

Question 3

Let R be the region in the first quadrant bounded by the y-axis and the graphs of $y = 4x - x^3 + 1$ and $y = \frac{3}{4}x$.



(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the x-axis.

(c) Write an expression involving one or more integrals that gives the perimeter of R. Do not evaluate.

Region R

$4x - x^3 + 1 = \frac{3}{4}x$ when $x = 1.94045 = A$

(a) Area $= \int_0^A \left(4x - x^3 + 1 - \frac{3}{4}x\right) dx$

$= 4.514$ or 4.515

3 { 1 : limits; 1 : integrand; 1 : answer }

(b) Volume

$= \pi \int_0^A \left(\left(4x - x^3 + 1\right)^2 - \left(\frac{3}{4}x\right)^2 \right) dx$

$= 18.291\pi$ or 57.463

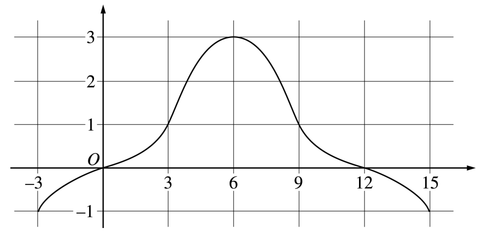
3 { 1 : limits and constant; 1 : integrand; 1 : answer }

(c) Perimeter $= 1 + \sqrt{(1.940)^2 + (1.455)^2}$

$+ \int_0^A \sqrt{1 + (4 - 3x^2)^2}\, dx$

3 { 1 : uses $y' = 4 - 3x^2$ in integrand; 1 : arc length integral; 1 : answer }

Question 4



Graph of f

The graph of a differentiable function f on the closed interval $[-3,15]$ is shown in the figure above. The graph of f has a horizontal tangent line at $x = 6$. Let $g(x) = 5 + \int_6^x f(t)\,dt$ for $-3 \le x \le 15$.

(a) Find $g(6)$, $g'(6)$, and $g''(6)$.

(b) On what intervals is g decreasing? Justify your answer.

(c) On what intervals is the graph of g concave down? Justify your answer.

(d) Find a trapezoidal approximation of $\int_{-3}^{15} f(t)\,dt$ using six subintervals of length $\Delta t = 3$.

(a) $g(6) = 5 + \int_6^6 f(t)\,dt = 5$

$g'(6) = f(6) = 3$

$g''(6) = f'(6) = 0$

3 { 1 : $g(6)$; 1 : $g'(6)$; 1 : $g''(6)$ }

(b) g is decreasing on $[-3,0]$ and $[12,15]$ since $g'(x) = f(x) < 0$ for $x < 0$ and $x > 12$.

3 { 1 : $[-3,0]$; 1 : $[12,15]$; 1 : justification }

(c) The graph of g is concave down on $(6,15)$ since $g' = f$ is decreasing on this interval.

2 { 1 : interval; 1 : justification }

(d) $\frac{3}{2}(-1 + 2(0 + 1 + 3 + 1 + 0) - 1)$
$= 12$

1 : trapezoidal method

Question 5

Consider the differential equation $\dfrac{dy}{dx} = \dfrac{3-x}{y}$.

(a) Let $y = f(x)$ be the particular solution to the given differential equation for $1 < x < 5$ such that the line $y = -2$ is tangent to the graph of f. Find the x-coordinate of the point of tangency, and determine whether f has a local maximum, local minimum, or neither at this point. Justify your answer.

(b) Let $y = g(x)$ be the particular solution to the given differential equation for $-2 < x < 8$, with the initial condition $g(6) = -4$. Find $y = g(x)$.

(a) $\dfrac{dy}{dx} = 0$ when $x = 3$

$$\left.\frac{d^2y}{dx^2}\right|_{(3,-2)} = \left.\frac{-y - y'(3-x)}{y^2}\right|_{(3,-2)} = \frac{1}{2},$$

so f has a local minimum at this point.

or

Because f is continuous for $1 < x < 5$, there is an interval containing $x = 3$ on which $y < 0$. On this interval, $\dfrac{dy}{dx}$ is negative to the left of $x = 3$ and $\dfrac{dy}{dx}$ is positive to the right of $x = 3$. Therefore f has a local minimum at $x = 3$.

3 $\begin{cases} 1 : x = 3 \\ 1 : \text{local minimum} \\ 1 : \text{justification} \end{cases}$

(b) $y\,dy = (3-x)\,dx$

$$\frac{1}{2}y^2 = 3x - \frac{1}{2}x^2 + C$$

$$8 = 18 - 18 + C\ ;\ C = 8$$

$$y^2 = 6x - x^2 + 16$$
$$y = -\sqrt{6x - x^2 + 16}$$

6 $\begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivative of } dy \text{ term} \\ 1 : \text{antiderivative of } dx \text{ term} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition } g(6) = -4 \\ 1 : \text{solves for } y \end{cases}$

Note: max 3/6 [1-1-1-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

Question 6

The Maclaurin series for $\ln\left(\dfrac{1}{1-x}\right)$ is $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$ with interval of convergence $-1 \le x < 1$.

(a) Find the Maclaurin series for $\ln\left(\dfrac{1}{1+3x}\right)$ and determine the interval of convergence.

(b) Find the value of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

(c) Give a value of p such that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges, but $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ diverges. Give reasons why your value of p is correct.

(d) Give a value of p such that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges, but $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges. Give reasons why your value of p is correct.

(a) $\ln\left(\dfrac{1}{1+3x}\right) = \ln\left(\dfrac{1}{1-(-3x)}\right)$

$$= \sum_{n=1}^{\infty} \frac{(-3x)^n}{n} \text{ or } \sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n} x^n$$

We must have $-1 \le -3x < 1$, so interval of convergence is $-\dfrac{1}{3} < x \le \dfrac{1}{3}$.

$2 \begin{cases} 1 : \text{series} \\ 1 : \text{interval of convergence} \end{cases}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \ln\left(\frac{1}{1-(-1)}\right) = \ln\left(\frac{1}{2}\right)$

1 : answer

(c) Some p such that $0 < p \le \dfrac{1}{2}$ because $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges by AST, but the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ diverges for $2p \le 1$.

$3 \begin{cases} 1 : \text{correct } p \\ 1 : \text{reason why } \sum \frac{(-1)^n}{n^p} \text{ converges} \\ 1 : \text{reason why } \sum \frac{1}{n^{2p}} \text{ diverges} \end{cases}$

(d) Some p such that $\dfrac{1}{2} < p \le 1$ because the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $p \le 1$ and the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges for $2p > 1$.

$3 \begin{cases} 1 : \text{correct } p \\ 1 : \text{reason why } \sum \frac{1}{n^p} \text{ diverges} \\ 1 : \text{reason why } \sum \frac{1}{n^{2p}} \text{ converges} \end{cases}$