Student Performance Q&A:

2002 AP® Calculus Free-Response Questions

The following comments are provided by the Chief Reader regarding the 2002 free-response questions for AP Calculus AB and AP Calculus BC. *They are intended to assist AP workshop consultants as they develop training sessions to help teachers better prepare their students for the AP Exams*. They give an overview of each question and its performance, including typical student errors. General comments regarding the skills and content that students frequently have the most problems with are included. Some suggestions for improving student performance in these areas are also included. Consultants are encouraged to use their expertise to create strategies for teachers to improve student performance in specific areas.

AB/BC Question 1

What was intended by the question?

This problem presented a region bounded between two graphs and two vertical lines. Students were asked to use both integration and differentiation to answer some straightforward questions about this region. Part (a) required the use of a definite integral to find the area of the region. In part (b), the region revolved about a horizontal line, resulting in a solid with cross sections in the shape of "washers," and the volume of the region was asked for, requiring another use of a definite integral. Students were expected to use the numerical integration capabilities of a graphing calculator to evaluate these definite integrals. Part (c) required the use of differentiation to find the absolute minimum height of the region.

How well did students perform?

Students scored fairly well on this problem with almost 7 percent of the AB students and 17 percent of the BC students receiving a 9. The mean score was 4.08 for the AB students indicating a reasonable overall performance. The mean score was 5.57 for the BC students, a very strong performance.

What were common errors or omissions?

Many students were able to find the area of the region but could not handle the integral for the volume because of the complication of rotating around a horizontal line that was not the x-axis. Some students thought that $y = 4$ was a vertical line.

In part (c) students were expected to use calculus and show the mathematical steps needed to analyze the location of the absolute minimum and maximum. Some students failed to consider where $h'(x) = 0$ to find the critical point, and some did not consider both endpoints as candidates.

Based on your experience at the AP Reading, what message would you like to send to teachers that could improve the performance of their students on the exam?

Students did not need to compute antiderivatives to compute the definite integrals in parts (a) and (b); the calculator may be used to calculate the value of the definite integral without further explanation once the setup of the definite integral is shown.

Part (c) created a difficult situation to grade because students could graph the function for the height of the region and read the location of the absolute maximum and absolute minimum off the graph without using any calculus knowledge. The grading standard tried to separate those students who appeared to use calculus from those who simply used graphing calculator technology, but that distinction was sometimes hard to make. When students are asked to show their analysis or justify an answer, they should include mathematical (noncalculator) reasons, not merely calculator results.

Have students show their work. When finding a critical point of a function, explicitly state the equation that is being solved for determining where the derivative is equal to 0. Do not expect a Reader to infer this equation from a graph or a sign chart. The student should explicitly indicate consideration of both endpoints and the critical point(s) when investigating an absolute maximum or absolute minimum over a closed interval.

Consult the "Free-Response Instruction Commentary" in the Teachers' Corner for Calculus AB or Calculus BC at AP Central (apcentral.collegeboard.com) for additional comments about the general instruction to "show all your work."

AB/BC Question 2

What was intended by the question?

This problem involved a "real-life" model of people entering and leaving an amusement park, and students were expected to interpret the meanings of their calculations in that context. In part (a), students needed to recognize that the function E represented the rate of accumulation of people entering the park, and hence that the total accumulation over a time interval could be obtained by a definite integral. Part (b) took the same idea a step further, asking students to calculate ticket revenues based on different pricing over two time intervals. Part (c) presented a function H defined in terms of a definite integral. The value of $H'(17)$ was easily computed, provided students recognized that the Fundamental Theorem of Calculus could be applied. More importantly, students were expected to interpret, in words, the meanings of both $H(17)$ and $H'(17)$. These interpretations played a role in part (d), which essentially asked students to find when the maximum value of the function H is achieved.

How well did students perform?

The mean score was 3.13 for the AB students and 4.89 for the BC students, indicating a fair performance by the AB population but a good performance by the BC group. Over 31 percent of the AB students earned no points on this problem, while about 12 percent of the BC students did so. About 7.7 percent of the BC students earned all 9 points while only 2.5 percent of the AB students were able to earn all 9 points.

What were common errors or omissions?

The most common error in part (a) was to simply evaluate $E(17)$ or $E(17)-L(17)$, or to attempt a discrete analysis without seeing the connection to a definite integral. Students who recognized the concepts being tested in parts (a) and (b) generally received at least 3 of the 4 possible points for those two parts.

The most common error in calculating $H'(17)$ was not recognizing the need to use the Fundamental Theorem of Calculus or not using the theorem correctly. A large number of students used $H'(17)=E'(17)-L'(17)$. The interpretations of $H(17)$ and $H'(17)$ were very hard for students to make. A number of students wrote the same interpretation for both items. Those that recognized the conceptual connection that $H(t)$ measured population in the park at time t and phrased $H(17)$ as a population had a much easier time with $H'(17)$.

Students could integrate by hand or with a CAS capable calculator to find a closed form expression for the number of people who entered the park and the number of people who left the park. This approach, however, was neither necessary nor efficient to handle the questions in this problem.

Based on your experience at the AP Reading, what message would you like to send to teachers that could improve the performance of their students on the exam?

One of the primary goals of the changes that AP Calculus has undergone in the past several years is to move away from testing rote manipulation and toward problems that probe understanding of the fundamental concepts. This problem was an example of one that required students to make connections between calculus and the "real" world, and between topics within calculus itself.

Have students show their work. Clearly show the setup for all definite integrals that must be evaluated. When finding a critical point of a function, explicitly state the equation that is being solved for determining where the derivative is equal to 0. Do not expect a Reader to infer this equation from a graph or a sign chart.

AB Question 3

What was intended by the question?

This problem presented the velocity and initial position of an object moving along the x-axis. Part (a) required knowledge of the relationship between velocity and its derivative, acceleration. Part (b) addressed the distinction between velocity and speed by presenting students with two statements that could appear contradictory, but in fact, are both true. The words "velocity" and "speed" are often used interchangeably in everyday language, but the technical distinction between the two is highlighted in calculus. Parts (c) and (d) also focused on this distinction. The total distance traveled by the object is calculated with a definite integral of speed (absolute value of velocity). Finding the position of the object at time $t=4$ involves displacement, calculated with a definite integral of velocity.

How well did students perform?

The mean score was 3.12 indicating a fair performance for the AB students. Only 1 percent of the approximately 154,000 AB students earned all 9 points, but 86 percent of the students did earn some points on this problem.

What were common errors or omissions?

Most students had little trouble with part (a). Part (b), however, was the most difficult to read. It was evident that many students had a poor understanding of the difference between velocity and speed. Poor writing and reasoning skills were evident in the answers to part (b). Very few students attempted to write and analyze an analytic expression for speed. Instead, many made observations based on the graph of the velocity and/or speed. This approach was allowed since the sine function was considered such a well-known function. Poor understanding of the change in direction in part (c) led many students to assume that the distance traveled was the same as the displacement. In addition, poor algebra skills were evident in part (c) and (d) where students who attempted to do the problems by hand made lots of mistakes in both algebra and arithmetic.

Based on your experience at the AP Reading, what message would you like to send to teachers that could improve the performance of their students on the exam?

Students who realized that parts (c) and (d) could be done with the calculator to evaluate definite integrals saved themselves much time and trouble. Students need to practice mathematical writing skills to help communicate their reasoning and explanations to the Reader.

AB/BC Question 4

What was intended by the question?

In this problem, students were given a graphical representation of a function *f*, and another function *g* that was defined in terms of a definite integral of *f*. While it was possible to find piecewise algebraic definitions for *f* and *g*, the questions asked were most efficiently and directly answered by the using the Fundamental Theorem of Calculus and reasoning based on the graph of *f*. Part (a) asked for calculations of $g(-1)$, $g'(-1)$, and $g''(-1)$. These values could be found using, respectively, an area, ordinate, and slope related to the graph of *f*. Using the fact that $f = g'$, part (b) required relating the sign of *f* (positive or negative) to the behavior of *g* (increasing or decreasing). Similarly, using the fact that $f' = g''$, part (c) required relating the behavior of the slope of the graph of *f* to the concavity of the graph of *g*. Part (d) asked for a sketch of the graph of *g*. Utilizing previous parts of the problem helped in determining characteristics of the graph.

How well did students perform?

The mean score was 2.87 for the AB students and 5.32 for the BC students. Almost 23 percent of the AB population and 7.4 percent of the BC population earned no points on this problem, a disappointment considering the emphasis placed on previous AP Calculus Exams on analyzing the behavior of a function given the graph of its derivative.

What were common errors or omissions?

Perhaps because this was the first problem in the non-calculator part of the exam, most students at least made an attempt at this problem. Though students generally did work in each part of the problem, most did their best work in parts (a) and (d). Having a lower limit of 0 in the definite integral for *f* caused errors in computing $g(-1)$ when the student did not realize the value was negative.

In parts (b) and (c), many students were able to correctly identify the interval where the function g was increasing and the interval where the graph of g was concave down. Not as many of these students were able to provide adequate reasons to support their conclusions. The reason points in parts (b) and (c) were only awarded if the student provided a connection between the functions f and g. Some students tried to justify their answer to part (b) by using an argument based on accumulating area. These arguments were problematic as they were typically vague or incomplete. It was rare for a student to earn the reason point going this route since the integral defining the function g had a lower limit of 0 and the student would have to accumulate area from -1 to 1.

Even students who did no correct work in parts (a) through (c) typically attempted to sketch the graph in part (d). A common error was to shift the correct graph up (or down) the y-axis, sometimes to be consistent with the incorrect computations in part (a).

Many students were troubled by the cusp at $x = 0$ on the graph of f and believed this caused g to be undefined at $x = 0$. Students with this misunderstanding could potentially make errors in each part of the problem, particularly in parts (b) and (d).

A small percentage of students attempted to solve the problem analytically.

Based on your experience at the AP Reading, what message would you like to send to teachers that could improve the performance of their students on the exam?

To enable their students to perform better on this sort of problem, teachers are urged to continue to emphasize the Fundamental Theorem of Calculus as well as a graphical approach to problem solving. Students should work with functions defined by definite integrals in which the lower limit is not always the left most endpoint of the interval of interest. Students continue to have difficulty properly justifying their conclusions and more experience with this would improve their performance on the exam. Students need to practice mathematical writing skills to help communicate their reasoning and explanations to the Reader.

AB Question 5

What was intended by the question?

This problem presented a common related rates setting with several variables (radius, depth, volume), related by geometry to the water evaporating in a conical container. Part (a) asked students to calculate the volume of water when its depth was $h = 5$ cm. The purpose of this part was to prompt students to establish the relationships among the radius, depth, and volume variables. Part (b) then asked students to relate the rate of change of the volume to the given rate of change of the depth of water. Part (c) introduced another related quantity, the exposed surface area, and asked students to verify a direct proportionality relationship between the rate of change of the volume and the exposed surface area of the water. The constant of proportionality in this case was precisely the given constant rate of change of the water's depth.

How well did students perform?

This was the first related rates problem on the free-response section in several years. The mean score was 2.29, indicating a poor performance by the AB students. About 41 percent of the students earned no points on this problem and less than 2 percent earned all 9 points.

What were common errors or omissions?

An overwhelming number of students were unable to compute the volume even though they were given the formula. They failed to develop the proper relationship between radius and height. The easiest way to do part (b) was to write a formula for the volume in terms of a single variable yet many students tried to solve this as a problem with two independent variables. They left themselves open to all kinds of mistakes since many failed to recognize the need for the product rule or failed to correctly use the chain rule to include both $\frac{dh}{dt}$ and $\frac{dr}{dt}$.

Part (c) was a very difficult section for students. Many did not understand what was meant by the phrase "…is directly proportional to…." Those who were successful most often would write down the formula for the area of the circle as well as their expression for $\frac{dV}{dt}$, then manipulate one expression or the other until they noticed the two expressions differed only by a constant.

Based on your experience at the AP Reading, what message would you like to send to teachers that could improve the performance of their students on the exam?

Students need to improve their ability to work with multiple variables, either by properly converting to a function with one independent variable or by applying the product and chain rules correctly. Students need to understand the difference between parameters and independent variables.

AB Question 6

What was intended by the question?

This problem presented data in tabular form for a function *f* and its derivative f', along with information about the sign of the second derivative. Part (a) required students to use the Fundamental Theorem of Calculus and some basic properties of integrals to calculate a specific definite integral. Part (b) asked for the calculation of a linear approximation using appropriate data from the table. Students needed to interpret the sign of the second derivative in terms of concavity and relate this information to the tangent line. Part (c) required students to recognize that the Mean Value Theorem could be applied to f' to show the existence of a real number c such that $f''(c) = 6$, the average rate of change of f' over the interval $[0, 0.5]$. Part (d) presented students with a piecewise algebraic expression for a function *g* that fit all of the given points on the graph of *f*. To determine that $f \neq g$, students needed to appeal to some inconsistency with the first or second derivatives of the respective functions.

How well did students perform?

The mean score was 2.33, indicating a poor performance by the AB students. It was difficult to earn all 9 points as only 0.3 percent of the students achieved this, and 30 percent of the students earned no points. Almost 92 percent of the students earned 5 points or less, mostly from just parts (a) and (b).

What were common errors or omissions?

Most students were able to correctly evaluate the integral in part (a) but those who could not either made no use of the Fundamental Theorem of Calculus or failed to antidifferentiate the constant term correctly. Students were able to earn the first two points in part (b) for computing a correct tangent line and finding the correct approximation at $x = 1.2$, but many were not able to provide a valid reason for the correct conclusion about the comparison between the approximation and the actual function value. In part (c), few students made explicit reference to the Mean Value Theorem. In part (d), students realized that there was a problem at $x = 0$ but very few were able to give a correct and complete reason for their conclusion. The most common error was to claim that $g'(0) = 1$.

Based on your experience at the AP Reading, what message would you like to send to teachers that could improve the performance of their students on the exam?

This is a good example of a problem that emphasizes the multirepresentational approach to calculus and the connections between functions represented in tabular and analytical form. Theory is still an important ingredient in the understanding of calculus and needs to be integrated into problem solving strategies. Students need to practice mathematical writing skills to help communicate their reasoning and explanations to the Reader.

BC Question 3

What was intended by the question?

This problem involving parametric equations described the motion of a roller coaster car. Both coordinates of the car's position were given. While the components of the velocity vector could be calculated directly, these were also stated in the problem. The emphasis in this problem was on using the velocity vector's components in a variety of ways to describe characteristics of the car's motion and its path at various times. Part (a) asked for the slope of the car's path at time $t = 2$, requiring students to formulate the evaluation of $\frac{dy}{dx}$ in terms of $\frac{dy}{dt}$ and $\frac{dx}{dt}$. Part (b) asked students to first determine the time at which the car is at a specific horizontal position, a calculation that required the numerical equation solver of a graphing calculator. Once this time was determined, students needed to use the relationship between the velocity and its derivative to determine the components of the acceleration vector. Part (c) also required the use of a graphing calculator to determine the times at which $y'(t) = 0$. At this instant, the speed of the car, given by the magnitude of the velocity vector, is $|x'(t)|$. Students may or may not have used a graphing calculator to find the times at which $y(t) = 0$ in part (d). In either case, an appropriate definite integral expression needed to be given whose evaluation would result in the average speed of the car over the interval defined by these times.

How well did students perform?

The mean score on this question was 3.03, reflecting a fair performance by the BC students. Only 1.7 percent of the students earned all 9 points while about 10 percent earned no points.

What were common errors or omissions?

In part (a) students were generally successful in being able to find the slope at a given point from the set of parametric equations. In part (b) most students recognized that the components of the velocity vector needed to be individually differentiated, but many students then went on to treat the derived components incorrectly by dividing them or by finding the length of the acceleration vector. Many students found it difficult to set up and/or compute an expression that represented the speed of the car at a particular time. The most common errors in part (d) were students being unable to find the two times on the given interval when the car was at ground level, and using an incorrect integrand in writing an expression for the average speed.

Parts (b), (c), and (d) asked students to compute the acceleration vector, speed, and average speed based on time values that also had to be computed from information given in the problem. Students needed to be careful to make sure the time values they decided to work with were within the domain specified in the stem of the problem.

A number of students reported that they had run out of time and had returned to this section of the exam without their calculators. This should not have affected their opportunity to earn all available points in part (d) where the equation was easily solved without the use of technology.

Based on your experience at the AP Reading, what message would you like to send to teachers that could improve the performance of their students on the exam?

The number of students who worked at least part of this problem with their calculator set to degree mode was higher than expected. Teachers are urged to remind their students to set their calculators to radian mode before entering the examination room. Teachers are also encouraged to reinforce the difference between vector values (such as a velocity vector) and scalar values (such as speed).

Have the students show their work. Explicitly state what equation the student is trying to solve, especially if this is being done with a calculator.

BC Question 5

What was intended by the question?

A differential equation was presented in this problem. The first two parts of the problem involved slope fields and Euler's method, which are Calculus BC-only topics. The last two parts addressed topics common to Calculus AB and Calculus BC. Part (a) asked for two solution curves sketched against a slope field provided for the differential equation. Part (b) took one of the initial conditions from part (a) and asked for a demonstration of the use of Euler's method to approximate another point on the solution curve. In part (c), students were asked to find the y-intercept of a linear solution to the differential equation. One of the two solution curves sketched in part (a) should be a straight line, so there was a strong visual clue for students to use in checking the reasonableness of their answers. The slope field also provided strong visual clues for part (d), but students needed to provide more than a visual explanation to justify that the solution curve passing through the origin has a local maximum there. The second derivative test was the most straightforward way to justify this result.

How well did students perform?

This question was split graded with parts (a) and (b) being BC-only material (slope fields and Euler's method) and parts (c) and (d) contributing to the Calculus AB subscore grade. The mean score on parts (a) and (b) was 2.42 (out of 4), with about 31 percent earning all 4 points. The mean score on the last two parts was 1.58 (out of 5), with only 2 percent earning all 5 points and 28 percent earning none of the 5 points. The total mean was 4.0 indicating a reasonably good performance.

What were common errors or omissions?

It can be difficult to get the precise solution curve when sketching on a slope field. Therefore the points in part (a) were awarded for the right "location," the right "extent," and the right "shape," with close calls going in the student's favor. Some students had difficulty because they seemed to be drawing their curves from left to right instead of working from the given initial point outward. There were many students in part (b) who did not know how to calculate the value of the derivative from the differential equation for use in Euler's method.

In part (c) it was important for the students to argue from the differential equation rather than just graphically. While many students determined that $b = 1$, few could present a complete, correct argument to earn both point.

It was extremely difficult to earn the justification points in part (d) via a first derivative test based on changes in the sign of the derivative of the function g. Students needed to make use of the differential equation to justify a conclusion, not just appeal to behavior discerned from the slope field.

Based on your experience at the AP Reading, what message would you like to send to teachers that could improve the performance of their students on the exam?

More attention should be given to the concept of a differential equation, emphasizing the fundamental fact that a solution to a differential equation is a function that satisfies the equation. In addition, students should be presented with many opportunities to use the second derivative test instead of the first derivative test, including cases where the second derivative test is much easier to use or the first derivative test is not applicable.

Have students show their work. Clearly indicate the steps involved in using Euler's method. When asked to justify a local extremum, show why all appropriate conditions about the first and/or second derivatives are satisfied. Explain in words the conclusion obtained from a first or second derivative test.

BC Question 6

What was intended by the question?

This problem presented students with an explicit Maclaurin series for a function f. In part (a), students were asked to determine its interval of convergence. Part (b) then asked students to derive a Maclaurin series for f' by manipulating the given series. Finally, in part (c), the evaluation of the Maclaurin series for f' at the specific value $x = -1/3$ resulted in a geometric series whose sum could be found by a simple calculation.

How well did students perform?

All BC examinations in recent years have had a problem on series, and student performances have generally been weak. The mean score on the series question this year was 3.89, which was better than the results on the 2000 and 2001 BC exams. Still, only 2.8 percent earned all 9 points and about 11 percent earned no points.

What were common errors or omissions?

Most students got a good start on part (a) with a ratio test setup, algebra work, and a computation of the limit. Some then made algebra mistakes, however, in solving the inequality for x. Students were explicitly asked to justify their answer for the interval of convergence. The majority knew that they needed to test the endpoints but had trouble earning the justification points. A common mistake occurred when students tried to use a direct comparison test for the right endpoint to compare their series with the harmonic series. The comparison goes the wrong way to conclude divergence. Many in fact failed to recognize that their series at the right endpoint was exactly the harmonic series.

In part (b), students usually stated the first four terms correctly but many failed to get the correct general term because of a chain rule error. In part (c) students tended to sum the first four terms only. It was essential that the final answer reflect the correct sum for the series indicated in part (b). If the four terms and the general term were inconsistent (or if there were no general term given), it was almost impossible for the student to earn the last point.

Based on your experience at the AP Reading, what message would you like to send to teachers that could improve the performance of their students on the exam?

Using an infinite series to find the value of a function is a concept that some students still need practice with.

Have students show their work. Use appropriate and correct mathematical notation to indicate the steps in a ratio test, particularly when taking the limit. Identify series by name where appropriate (e.g., the harmonic series or a geometric series).

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