Question 2

An object moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time t, where

$$\frac{dx}{dt} = \tan\left(e^{-t}\right) \text{ and } \frac{dy}{dt} = \sec\left(e^{-t}\right)$$

for $t \geq 0$. At time $t = 1$, the object is at position $(2, -3)$.

(a) Write an equation for the line tangent to the curve at position $(2, -3)$.

(b) Find the acceleration vector and the speed of the object at time $t = 1$.

(c) Find the total distance traveled by the object over the time interval $1 \leq t \leq 2$.

(d) Is there a time $t \geq 0$ at which the object is on the y-axis? Explain why or why not.

(a) $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sec\left(e^{-t}\right)}{\tan\left(e^{-t}\right)} = \frac{1}{\sin\left(e^{-t}\right)}$$

$$\left.\frac{dy}{dx}\right|_{(2,\,-3)} = \frac{1}{\sin\left(e^{-1}\right)} = 2.780 \text{ or } 2.781$$

$$y + 3 = \frac{1}{\sin\left(e^{-1}\right)}(x - 2)$$

$2: \begin{cases} 1: \left.\dfrac{dy}{dx}\right|_{(2,\,-3)} \\ 1: \text{equation of tangent line} \end{cases}$

(b) $x''(1) = -0.42253, \quad y''(1) = -0.15196$

$a(1) = \langle -0.423, -0.152 \rangle$ or $\langle -0.422, -0.151 \rangle$.

$$\text{speed} = \sqrt{\left(\sec\left(e^{-1}\right)\right)^2 + \left(\tan\left(e^{-1}\right)\right)^2} = 1.138 \text{ or } 1.139$$

$2: \begin{cases} 1: \text{acceleration vector} \\ 1: \text{speed} \end{cases}$

(c) $$\int_1^2 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = 1.059$$

$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

(d) $$x(0) = x(1) - \int_0^1 x'(t)\, dt = 2 - 0.775553 > 0$$

The particle starts to the right of the y-axis.
Since $x'(t) > 0$ for all $t \geq 0$, the object is always moving to the right and thus is never on the y-axis.

$3: \begin{cases} 1: x(0) \text{ expression} \\ 1: x'(t) > 0 \\ 1: \text{conclusion and reason} \end{cases}$

2A

Work for problem 2(a)

$\frac{dy}{dx}$ = slope = $\frac{dy/dt}{dx/dt}$ —① . P (which is moving) is at $x = (2, -3)$ at $t = 1$.

$\therefore \left.\frac{dx}{dt}\right|_{t=1} = \tan(1/e)$ $\quad \left.\frac{dy}{dt}\right|_{t=1} = \sec(1/e)$

$\therefore$ slope $= \frac{\sec(1/e)}{\tan(1/e)}$ (from ①) $= \frac{1}{\sin(1/e)} = 2.780$

$\therefore$ Equation of tangent is $\boxed{y+3 = 2.780\,(x-2)}$

Work for problem 2(b)

$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}(\tan e^{-t}) = [\sec^2(e^{-t})] - e^{-t}$

Similarly, $\frac{d^2y}{dt^2} = \sec(e^{-t}) \tan(e^{-t}) . -e^{-t}$

at $t = 1$, acceleration vector $= \left[-\sec^2(1/e)\,e^{-1}, -e^{-1}\sec(1/e)\tan(1/e)\right]$

$= \boxed{(-0.422, -0.151)}$

speed$\left.\right|_{t=1} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(\tan^2(1/e) + \sec^2(1/e))}$

$= \boxed{1.138}$

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2A

Work for problem 2(c)

$$\text{distance travelled} = \int_{t_1}^{t_2} \text{speed}\,.dt$$

$$D \text{ from } t = 1 \to 2 \quad = \int_1^2 \sqrt{[(\tan^2(e^{-t}) + (\sec^2(e^{-t})]}\,.dt$$

$$= \boxed{1.059}$$

Work for problem 2(d)

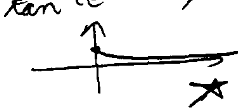
If the object (P) is on y-axis at some time, then $x = 0$ at that time, say $t = t^*$

$$\therefore \quad x - 2 = \int_1^t \tan(e^{-t})\,.dt \qquad \left[\Delta x = \int_1^2 dx\right]$$

If $x = 0$

$$\Rightarrow \quad -2 = \int_1^{t^*} \tan(e^{-t})\,.dt$$

[The curve of $\tan(e^{-t})$, $t > 0$ is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$\therefore$, However, since $e^{-t} > 0$ for $t > 0$ and tends to 0 as $t \to \infty$, we've that $\tan(e^{-t}) > 0$ $\forall t \geq 0$

$\therefore$ The area under the curve $\tan(e^{-t})$ for $t > 0$ from $t = 1$ to $t = t^*$ is always > 0.

Hence, there is no $t^* > 0$ for which P is on the y-axis.

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Work for problem 2(a)

at time $t=1$, $\frac{dx}{dt} = \tan(e^{-1})$, $\frac{dy}{dt} = \sec(e^{-1})$

$\therefore \frac{dy}{dx} = \frac{\sec(e^{-1})}{\tan(e^{-1})} = \frac{1}{\sin(e^{-1})} = 2.78$

$\therefore$ eq. of line: $y+3 = 2.78(x-2)$

Work for problem 2(b)

~~accel~~ $\text{acceleration} = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}\right)$

$= \left(\frac{-e^{-t}}{(\cos(e^{-t}))^2}, \frac{-e^{-t}\cdot\sin(e^{-t})}{(\cos(e^{-t}))^2}\right)$

$= (-0.42, -0.15)$

$\text{speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\Bigg|_{t=1} = \frac{\sqrt{(\sin(e^{-1}))^2+1}}{\cos(e^{-1})} = 1.14$

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Work for problem 2(c)

$$\text{distance} = \int_1^2 \left(\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\right) dt$$

$$= 1.06$$

Work for problem 2(d)

when the object is on the y-axis,

the x-coordinate of the object is 0.

the initial x-coordinate is 2,

$\frac{dx}{dt} > 0$ for all $t \geq 0$.

$\therefore$ (x-coordinate) > 0 for all $t \geq 0$.

$\therefore$ the object is not on the y-axis for all $t \geq 0$.

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2c

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Work for problem 2(a)

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx} = \frac{\sec(e^{-t})}{\tan(e^{-t})}$$

$$@\ t=1,\ \frac{dy}{dx} = 2.78058$$

$$y = 2.78058x + b$$

$$-3 = 2.78058(2) + b$$

$$b = -8.56115$$

$$\boxed{y = 2.781x - 8.561}$$

Work for problem 2(b)

$$\text{Speed} = \frac{dy}{dx}\ @\ t=1 = \frac{\sec(e^{-1})}{\tan(e^{-1})} = \boxed{2.781}$$

$$\frac{d^2x}{dt^2} = \frac{-e^{-t}}{(\cos(e^{-t}))^2}\ @\ t=1 = -0.4225$$

$$\frac{d^2y}{dt^2} = \frac{-e^{-t}\sin e^{-t}}{(\cos(e^{-t}))^2}\ @\ t=1 = -0.1519$$

$$\boxed{\text{acceleration } \vec{v} = (-0.423, -0.152)}$$

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2c

Work for problem 2(c)

$$\int_1^2 \frac{\sec(e^{-t})}{\tan(e^{-t})} = \boxed{4.710}$$

Work for problem 2(d)

Object on y-axis means $x=0$

$$x(t) = \int \frac{dx}{dt} = \int \tan(e^{-t}) =$$

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AP® CALCULUS BC
2006 SCORING COMMENTARY (Form B)

Question 2

Overview

This problem dealt with particle motion in the plane. Students were given the rate of change of the *x*- and *y*-coordinates as functions of time and the initial position of the particle at time $t = 1.$ Part (a) asked for the equation of the line tangent to the curve at the point corresponding to time $t = 1.$ Part (b) asked for the acceleration vector and speed of the object at time $t = 1.$ Part (c) tested if students could use a definite integral to compute the total distance traveled by the object over a specified time interval. For part (d) students needed to make use of the initial position and sign of the first derivative for the *x*-coordinate to deduce that the object could never be on the *y*-axis. This could be done by arguing about the increasing behavior of $x(t)$ or by using the Fundamental Theorem of Calculus to express $x(t)$ in terms of a definite integral that always took positive values.

Sample: 2A
Score: 8

The student earned 8 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and 2 points in part (d). In part (b) it was not necessary for the student to compute the exact second derivatives since the values at $t = 1$ could be computed numerically on the calculator. In part (d) the student's argument is only valid for $t^* > 1.$ If $t^* < 1,$ then the student's definite integral will be negative and might equal $-2.$ The student could have completed the reasoning and earned the last point by observing that when $t^* = 0,$ the value of the definite integral is only $-0.77555.$

Sample: 2B
Score: 6

The student earned 6 points: 2 points in part (a), 1 point in part (b), 2 points in part (c), and 1 point in part (d). The work in part (a) earned both points. The acceleration vector in part (b) is computed correctly, but the answers are only reported to two decimal places rather than the three decimal places specified in the exam instructions. The student therefore did not earn the first point. The speed is also computed correctly. Since a point already had been lost for a decimal presentation error, the student was not penalized again and earned the second point in part (b). For a similar reason, the student also earned both points in part (c). In part (d) the student earned the second point for observing that $\frac{dx}{dt} > 0$ for all $t \geq 0.$ However, the student never considers the value of $x(0)$ and the reasoning is not complete, so the other 2 points were not earned.

Sample: 2C
Score: 3

The student earned 3 points: 2 points in part (a) and 1 point in part (b). The work in part (a) earned both points. The student avoids the possibility of premature rounding by using intermediate calculations to five decimal places and only rounding at the final step. In part (b) the acceleration vector is correct, but the student believes the speed is found from the slope at $t = 1.$ In part (c) the student continues to use the expression for the slope and does not earn any points. In part (d) the student sets up an integral to try to compute the position $x(t)$ but is unable to complete this approach and earned no points.