Question 6

The function f is defined by $f(x) = \dfrac{1}{1+x^3}$. The Maclaurin series for f is given by

$$1 - x^3 + x^6 - x^9 + \cdots + (-1)^n x^{3n} + \cdots,$$

which converges to $f(x)$ for $-1 < x < 1$.

(a) Find the first three nonzero terms and the general term for the Maclaurin series for $f'(x)$.

(b) Use your results from part (a) to find the sum of the infinite series $-\dfrac{3}{2^2} + \dfrac{6}{2^5} - \dfrac{9}{2^8} + \cdots + (-1)^n \dfrac{3n}{2^{3n-1}} + \cdots$.

(c) Find the first four nonzero terms and the general term for the Maclaurin series representing $\int_0^x f(t)\,dt$.

(d) Use the first three nonzero terms of the infinite series found in part (c) to approximate $\int_0^{1/2} f(t)\,dt$. What are the properties of the terms of the series representing $\int_0^{1/2} f(t)\,dt$ that guarantee that this approximation is within $\dfrac{1}{10{,}000}$ of the exact value of the integral?

(a) $f'(x) = -3x^2 + 6x^5 - 9x^8 + \cdots + 3n(-1)^n x^{3n-1} + \cdots$	$2: \begin{cases} 1: \text{first three terms} \\ 1: \text{general term} \end{cases}$
(b) The given series is the Maclaurin series for $f'(x)$ with $x = \frac{1}{2}$. $f'(x) = -\left(1+x^3\right)^{-2}\left(3x^2\right)$ Thus, the sum of the series is $f'\left(\frac{1}{2}\right) = -\dfrac{3\left(\frac{1}{4}\right)}{\left(1+\frac{1}{8}\right)^2} = -\dfrac{16}{27}$.	$2: \begin{cases} 1: f'(x) \\ 1: f'\left(\frac{1}{2}\right) \end{cases}$
(c) $\displaystyle\int_0^x \frac{1}{1+t^3}\,dt = x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10} + \cdots + \frac{(-1)^n x^{3n+1}}{3n+1} + \cdots$	$2: \begin{cases} 1: \text{first four terms} \\ 1: \text{general term} \end{cases}$
(d) $\displaystyle\int_0^{1/2} \frac{1}{1+t^3}\,dt \approx \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^4}{4} + \frac{\left(\frac{1}{2}\right)^7}{7}$. The series in part (c) with $x = \frac{1}{2}$ has terms that alternate, decrease in absolute value, and have limit 0. Hence the error is bounded by the absolute value of the next term. $\displaystyle\left\lvert \int_0^{1/2} \frac{1}{1+t^3}\,dt - \left(\frac{1}{2} - \frac{\left(\frac{1}{2}\right)^4}{4} + \frac{\left(\frac{1}{2}\right)^7}{7}\right) \right\rvert < \frac{\left(\frac{1}{2}\right)^{10}}{10} = \frac{1}{10240} < 0.0001$	$3: \begin{cases} 1: \text{approximation} \\ 1: \text{properties of terms} \\ 1: \text{absolute value of} \\ \quad\ \text{fourth term} < 0.0001 \end{cases}$

NO CALCULATOR ALLOWED

Do not write beyond this border.

Work for problem 6(a)

$f'(x) = \boxed{-3x^2 + 6x^5 - 9x^8 + \ldots + (-1)^n(3n)x^{3n-1} + \ldots}$

Work for problem 6(b)

$\therefore \boxed{x = \frac{1}{2}}$ for this series, if we let $x = \frac{1}{2}$, we get:

$\frac{-3}{2^2} + \frac{6}{2^5} - \frac{9}{2^8} + \ldots + \frac{(-1)^n(3n)}{2^{3n-1}} + \ldots$

this is the derivative of $f'(x)$ at $x = \frac{1}{2}$

$f'(x) = \frac{-3x^2}{(1+x^3)^2}$

$f'\left(\frac{1}{2}\right) = \frac{-3\left(\frac{1}{2}\right)^2}{\left(1+\left(\frac{1}{2}\right)^3\right)^2}$ $\frac{-3\left(\frac{1}{4}\right)}{\left(1+\frac{1}{8}\right)^2} = \frac{\frac{-3}{4}}{\left(\frac{9}{8}\right)^2} = \frac{\frac{-3}{4}}{\frac{81}{64}} = \boxed{\frac{-3(64)}{4(81)}}$ sum of the series

Continue problem 6 on page 15.

Work for problem 6(c)

$$\int_0^x f(t)\,dt = x - \frac{1}{4}x^4 + \frac{1}{7}x^7 - \frac{1}{10}x^{10} + \dots + \frac{(-1)^n x^{3n+1}}{(3n+1)} + \dots - (0)$$

$$\int_0^x f(t)\,dt = x - \frac{1}{4}x^4 + \frac{1}{7}x^7 - \frac{1}{10}x^{10} + \dots + \frac{(-1)^n x^{3n+1}}{(3n+1)} + \dots$$

Work for problem 6(d)

$$\int_0^{1/2} f(t)\,dt = \left[x - \frac{1}{4}x^4 + \frac{1}{7}x^7\right]_0^{1/2}$$

$$= \frac{1}{2} - \frac{1}{4}\cdot\frac{1}{16} + \frac{1}{7}\cdot\frac{1}{128} - (0)$$

$$= \boxed{\frac{1}{2} - \frac{1}{64} + \frac{1}{896}}$$

This is an alternating series, so the error is less than or equal to the next unused term

$$\text{error} \le \frac{1}{10}\left(\frac{1}{2^{10}}\right) = \frac{1}{10}\cdot\frac{1}{1024} = \boxed{\frac{1}{10240}}$$

This is the error, and it is less than $\frac{1}{10000}$, thus the integral done here is within $\frac{1}{10000}$ of the exact integral

NO CALCULATOR ALLOWED

Work for problem 6(a)

$f'(x) = -3x^2 + 6x^5 - 9x^8$

Work for problem 6(b)

$-\frac{3}{2^2} + \frac{6}{2^5} - \frac{9}{2^8} + \dots + (-1)^n \frac{3n}{2^{3n-1}} = f'(\frac{1}{2})$

$f'(x) = \frac{-3x^2}{(1+x^3)^2}$ $\therefore f'(\frac{1}{2}) = \frac{-3(\frac{1}{4})}{(\frac{9}{8})^2} = \frac{-3}{4} \div \frac{81}{64}$

$= \frac{-3}{4} \times \frac{64}{81} = -\frac{16}{27}$

Do not write beyond this border.

Do not write beyond this border.

Continue problem 6 on page 15.

Work for problem 6(c)

$$\int_0^x f(t) = \int_0^x 1 - x^3 + x^6 - x^9 \ldots \, dt$$

$$= x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10} \ldots$$

$\therefore$ first four non zero term $\int_0^x f(t)$ is $x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10}$

Work for problem 6(d)

$$\int_0^{1/2} f(t) = \frac{1}{2} - \frac{1}{4(2)^4} + \frac{1}{7(2)^7} = \frac{1}{2} - \frac{1}{64} + \frac{1}{896} = \frac{448}{896} - \frac{14}{896} + \frac{1}{896}$$

$$= \frac{435}{896}$$

because the absolute of 4th non zero term is less than $\frac{1}{10000}$

(test the error)

4th term $= \left|\frac{1}{2^{10} \times 10}\right| = \frac{1}{10240} \Rightarrow$ less than $\frac{1}{10000}$

Do not write beyond this border.

Work for problem 6(a)

$f(x) = 1 - x^3 + x^6 - x^9 + \cdots + (-1)^n x^{3n}$

$f'(x) = -3x^2 + 6x^5 - 9x^8 + \cdots + 3n(-1)^n x^{(3n-1)} + \cdots$

(first 3)

Work for problem 6(b)

When $x = \frac{1}{2}$.

(a) ~~series~~ meets (b)'s series.

$f'(\frac{1}{2}) = -\frac{3}{2^2} + \frac{6}{2^5} - \frac{9}{2^8} + \cdots + (-1)^n \frac{3n}{2^{3n-1}}$.

$= -\frac{3}{4} + \frac{6}{32} - \frac{9}{256} + \cdots$

$\times(-\frac{1}{4})$ $\times(-\frac{3}{16})$

$\times 3$

Do not write beyond this border.

Continue problem 6 on page 15.

Work for problem 6(c)

$\int_0^x f(t)\,dt$

$= x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^9}{9} + \cdots + (-1)^n \cdot \frac{x^{3n+1}}{3n+1} + \cdots$

Work for problem 6(d)

$\int_0^{\frac{1}{2}} f(t)\,dt$.

because this series is decreasing alternating series,.

$\left|\left(\int_0^{\frac{1}{2}} f(t)\,dt\right) - \left[x - \frac{x^4}{4} + \frac{x^7}{7}\right]_0^{\frac{1}{2}}\right| \leqq \underline{\left|\left[-\frac{x^9}{9}\right]_0^{\frac{1}{2}}\right|}$.

↳ $\underline{\underline{\frac{1}{512 \times 9} < \frac{1}{10000}}}$.

Question 6

Overview

This problem presented students with the Maclaurin series for $f(x) = \frac{1}{1+x^3}$. In part (a) students were asked to find the first three nonzero terms and the general term for the Maclaurin series for $f'(x)$. In part (b) they then needed to use their results to find the sum of the infinite series that is obtained by evaluating the Maclaurin series for $f'(x)$ at $x = \frac{1}{2}$. Part (c) asked for the first four nonzero terms and the general term for the Maclaurin series for $\int_0^{1/2} f(t)\,dt$, and then part (d) asked students to use the first three nonzero terms to approximate this definite integral. The second question in part (d) asked students to list the properties of the terms of the series representing this definite integral that guaranteed that their approximation is within $\frac{1}{10{,}000}$ of the exact value of the integral. This question tested whether students knew that the terms must not only alternate and converge to 0, but must also decrease in absolute value to use the error bound for alternating series.

Sample: 6A
Score: 8

The student earned 8 points: 2 points in each of parts (a), (b), (c), and (d). The student correctly reports the first three nonzero terms and the general term in the Maclaurin series for $f'(x)$ in part (a). In part (b) the derivative $f'(x)$ and the special value $f'\left(\frac{1}{2}\right)$ are both computed correctly. In part (c) the student correctly reports the first four nonzero terms and the general term of the Maclaurin series for $\int_0^x f(t)\,dt$. In part (d) the student finds the appropriate approximation. The student should have used an approximation symbol, but the inappropriate use of the equal sign did not lose the first point. The sum does not need to be simplified. The only property of the terms that the student gives is that the series is alternating, and this is not enough to conclude that the approximation is within the given accuracy. The student earned the third point for establishing an error bound that is less than $\frac{1}{10{,}000}$.

Sample: 6B
Score: 6

This student earned 6 points: 1 point in part (a), 2 points in part (b), 1 point in part (c), and 2 points in part (d). In part (a) the first three nonzero terms of the Maclaurin series for $f'(x)$ are correct, but the general term is not provided. The computation of $f'(x)$ and evaluation of $f'\left(\frac{1}{2}\right)$ are correct, which earned all points in part (b). The first four nonzero terms in the Maclaurin series for $\int_0^x f(t)\,dt$ are correct in part (c), but the general term is not provided. For part (d) the first three nonzero terms of the Maclaurin series for $\int_0^{1/2} f(t)\,dt$ are correctly summed,

Question 6 (continued)

and the absolute value of the fourth nonzero term is shown to be smaller than $\frac{1}{10{,}000}$. However, there is no mention of the properties of the terms of the series that support the use of this error bound.

Sample: 6C
Score: 3

This student earned 3 points: 2 points in part (a) and 1 point in part (c). The work in part (a) is correct. In part (b) the student recognizes that the series is the value of the derivative at $x = \frac{1}{2}$ but attempts to approximate this using the first three terms of the series rather than compute the derivative directly and evaluate it at $x = \frac{1}{2}$. In part (c) the student makes an antidifferentiation error in the fourth term and did not earn the first point. The general term is correct, however, and earned the second point. In part (d) the student never finds the approximation using the first three terms. The student cites two properties of the terms but fails to include the property that the terms converge to zero and thus did not earn the second point. The reported error bound is actually greater than $\frac{1}{10{,}000}$, so the student did not earn the last point.