Question 3

An object moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time t, where

$$\frac{dx}{dt} = \sin^{-1}\left(1 - 2e^{-t}\right) \text{ and } \frac{dy}{dt} = \frac{4t}{1 + t^3}$$

for $t \geq 0$. At time $t = 2$, the object is at the point $(6, -3)$. (Note: $\sin^{-1} x = \arcsin x$)

(a) Find the acceleration vector and the speed of the object at time $t = 2$.

(b) The curve has a vertical tangent line at one point. At what time t is the object at this point?

(c) Let $m(t)$ denote the slope of the line tangent to the curve at the point $(x(t), y(t))$. Write an expression for $m(t)$ in terms of t and use it to evaluate $\lim_{t\to\infty} m(t)$.

(d) The graph of the curve has a horizontal asymptote $y = c$. Write, but do not evaluate, an expression involving an improper integral that represents this value c.

(a) $a(2) = \langle 0.395 \text{ or } 0.396, -0.741 \text{ or } -0.740\rangle$ Speed $= \sqrt{x'(2)^2 + y'(2)^2} = 1.207$ or 1.208	$2: \begin{cases} 1: \text{acceleration} \\ 1: \text{speed} \end{cases}$
(b) $\sin^{-1}\left(1 - 2e^{-t}\right) = 0$ $1 - 2e^{-t} = 0$ $t = \ln 2 = 0.693$ and $\frac{dy}{dt} \neq 0$ when $t = \ln 2$	$2: \begin{cases} 1: x'(t) = 0 \\ 1: \text{answer} \end{cases}$
(c) $m(t) = \frac{4t}{1+t^3} \cdot \frac{1}{\sin^{-1}\left(1 - 2e^{-t}\right)}$ $\lim_{t\to\infty} m(t) = \lim_{t\to\infty}\left(\frac{4t}{1+t^3} \cdot \frac{1}{\sin^{-1}\left(1 - 2e^{-t}\right)}\right)$ $= 0\left(\frac{1}{\sin^{-1}(1)}\right) = 0$	$2: \begin{cases} 1: m(t) \\ 1: \text{limit value} \end{cases}$
(d) Since $\lim_{t\to\infty} x(t) = \infty$, $c = \lim_{t\to\infty} y(t) = -3 + \int_2^{\infty} \frac{4t}{1+t^3}\, dt$	$3: \begin{cases} 1\text{: integrand} \\ 1\text{: limits} \\ 1\text{: initial value consistent} \\ \quad \text{with lower limit} \end{cases}$

Do not write beyond this border.

Work for problem 3(a)

speed: $\frac{dx}{dt} = \arcsin(1 - 2e^{-2}) = .817$

$\frac{dy}{dt} = \frac{4(2)}{1+2^3} = \frac{8}{9}$

$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$v = 1.208$

accel. vector: $\frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{1}{\sqrt{e^t - 1}}$

$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{-4(2t^3-1)}{(t^3+1)^2}$

$t=2$

$\langle .396, -.741 \rangle$

Work for problem 3(b)

vert tang: $\frac{dy}{dx}$ when $dx = 0$, $t = ?$

$0 = \arcsin(1 - 2e^{-t})$

$t = .693$

Continue problem 3 on page 9.

Do not write beyond this border.

Work for problem 3(c)

$$m(t) = \frac{4t}{(1+t^3)\arcsin(1-2e^{-t})}$$

$$\lim_{t\to\infty} \frac{4t}{(1+t^3)\arcsin(1-2e^{-t})} = 0$$

Work for problem 3(d)

$$\int_2^\infty \frac{4t}{1+t^3}\,dt - 3$$

Do not write beyond this border.

END OF PART A OF SECTION II
IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

Do not write beyond this border.

Work for problem 3(a)

acceleration vector = $(.396, -.741)$

Speed = 1.208

Work for problem 3(b)

Verticle tangent where $\frac{dx}{dt} = 0$

$\frac{dx}{dt} = 0$ at $t = .693$

Continue problem 3 on page 9.

Work for problem 3(c)

$$m(t) = \frac{4t}{(1+t^3)\sin^{-1}(1-2e^{-t})}$$

$$\lim_{t\to\infty} m(t) = 1$$

Work for problem 3(d)

$$\int_1^\infty \frac{4t}{(1+t^3)\sin^{-1}(1-2e^{-t})}\,dt$$

Do not write beyond this border.

Do not write beyond this border.

END OF PART A OF SECTION II
IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

Work for problem 3(a)

$v_x = \sin^{-1}(1-2e^{-t})$

$v_x(2) = .817$

$v_y = \frac{4t}{1+t^3}$ $\frac{1}{(1+t^3)^{-1}}$

$V_y(t) = .889$

$v = \langle .817, .889 \rangle$

$a_x = \frac{1-2e^{-t}}{\sqrt{1-(1-2e^{-t})^2}}$

$a_x(2) = 1.066$

$a_y = (4t \cdot -1(1+t^3)^{-2} \cdot 3t^2) + (4 \cdot \frac{1}{1+t^3})$

$a_y(2) = -.741$

$a = \langle 1.066, -.741 \rangle$

Work for problem 3(b)

$t = 1$

Continue problem 3 on page 9.

Do not write beyond this border.

Do not write beyond this border.

Work for problem 3(c)

$$m(t) = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx}$$

$$m(t) = \frac{\frac{4t}{1+t^3}}{\sin^{-1}(1-2e^{-t})}$$

$$\lim_{t\to\infty} m(t) = 0$$

Work for problem 3(d)

$$\int_0^{\infty} \frac{\frac{4t}{1+t^3}}{\sin^{-1}(1-2e^{-t})}$$

$$= \lim_{b\to\infty} \int_0^b \frac{\frac{4t}{1+t^3}}{\sin^{-1}(1-2e^{-t})}\, dt$$

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

Do not write beyond this border.

Do not write beyond this border.

Question 3

Overview

This problem dealt with particle motion in the plane. Students were given the rate of change of the x- and y-coordinates as functions of time and the initial position of a particle at time $t = 2.$ Part (a) asked for the acceleration vector and speed of the object at time $t = 2.$ Parts (b) and (c) dealt with lines tangent to the curve along which the object moves. In part (b) students had to find the time at which the curve has a vertical tangent line, and in part (c) students had to find a general expression for the slope of the line tangent to the curve at an arbitrary point on the curve. Students were also asked to evaluate the limit of this slope as $t \to \infty$. Part (d) tested the students' ability to use the Fundamental Theorem of Calculus to write an improper integral that represented the value of the horizontal asymptote for the graph of the curve.

Sample: 3A
Score: 9

The student earned all 9 points. Note that because this is a calculator-active question, the student was not required to show work leading to the answers presented in parts (a), (b), and (c). In particular, analytic expressions for the second derivatives were not required to earn the point for the acceleration vector in part (a).

Sample: 3B
Score: 6

The student earned 6 points: 2 points in part (a), 2 points in part (b), 1 point in part (c), and 1 point in part (d). In part (a) the student correctly presents an acceleration vector and a value for speed at time $t = 2$ to three decimal places. The student earned both points for part (a). In part (b) the student sets $x'(t) = 0$ and earned the first point. The student goes on to solve correctly for t to three decimal places and earned the second point. In part (c) the student presents a correct expression for $m(t)$ in terms of t and earned the first point. The limit is incorrectly evaluated, and the student did not earn the second point. In part (d) the student presents $m(t)$ as the integrand and did not earn the integrand point. The student presents limits of integration of the form $a \geq 0$ for the lower limit and infinity for the upper limit and earned the limits point. The student does not consider an initial condition and therefore did not earn the third point.

Sample: 3C
Score: 3

The student earned 3 points: 2 points in part (c) and 1 point in part (d). In part (a) the student presents analytic expressions for the x- and y-components of the acceleration vector. The incorrect expression in the numerator of the x-component of the acceleration vector leads to an incorrect numerical value in the presentation of the acceleration vector. The student did not earn the first point. The velocity vector at $t = 2$ is given but the speed is not calculated. The student did not earn the second point. In part (b) no equation is given and an incorrect t value is presented. The student did not earn either point. In part (c) the student gives a correct expression for $m(t)$ in terms of t and earned the first point. The limit is then correctly evaluated, and the student earned the second point. Note that the student was not required to show justification to earn the limit evaluation point. In part (d) the student presents an integral with the incorrect integrand and did not earn the first point. The student presents

Question 3 (continued)

limits of integration of the form $a \geq 0$ for the lower limit and infinity for the upper limit and earned the second point. Note that the use of the limit notation was not required to earn the second point. The student does not consider an initial condition and therefore did not earn the third point.