AP® CALCULUS BC
2006 SCORING GUIDELINES

Question 5

Consider the differential equation $\frac{dy}{dx} = 5x^2 - \frac{6}{y-2}$ for $y \neq 2$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(-1) = -4$.

(a) Evaluate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $(-1, -4)$.

(b) Is it possible for the x-axis to be tangent to the graph of f at some point? Explain why or why not.

(c) Find the second-degree Taylor polynomial for f about $x = -1$.

(d) Use Euler's method, starting at $x = -1$ with two steps of equal size, to approximate $f(0)$. Show the work that leads to your answer.

(a) $\left.\frac{dy}{dx}\right|_{(-1,\,-4)} = 6$

$$\frac{d^2y}{dx^2} = 10x + 6(y-2)^{-2}\frac{dy}{dx}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(-1,\,-4)} = -10 + 6\frac{1}{(-6)^2}6 = -9$$

$$3 : \begin{cases} 1 : \left.\frac{dy}{dx}\right|_{(-1,\,-4)} \\ 1 : \frac{d^2y}{dx^2} \\ 1 : \left.\frac{d^2y}{dx^2}\right|_{(-1,\,-4)} \end{cases}$$

(b) The x-axis will be tangent to the graph of f if $\left.\frac{dy}{dx}\right|_{(k,\,0)} = 0.$

The x-axis will never be tangent to the graph of f because $\left.\frac{dy}{dx}\right|_{(k,\,0)} = 5k^2 + 3 > 0$ for all k.

$$2 : \begin{cases} 1 : \frac{dy}{dx} = 0 \text{ and } y = 0 \\ 1 : \text{answer and explanation} \end{cases}$$

(c) $P(x) = -4 + 6(x+1) - \frac{9}{2}(x+1)^2$

$$2 : \begin{cases} 1 : \text{quadratic and centered at } x = -1 \\ 1 : \text{coefficients} \end{cases}$$

(d) $f(-1) = -4$

$$f\left(-\frac{1}{2}\right) \approx -4 + \frac{1}{2}(6) = -1$$

$$f(0) \approx -1 + \frac{1}{2}\left(\frac{5}{4} + 2\right) = \frac{5}{8}$$

$$2 : \begin{cases} 1 : \text{Euler's method with 2 steps} \\ 1 : \text{Euler's approximation to } f(0) \end{cases}$$

NO CALCULATOR ALLOWED

5A

Work for problem 5(a)

$$\frac{dy}{dx} = 5x^2 - \frac{6}{y-2}\Big|_{(-1,-4)} = 5(-1)^2 - \frac{6}{-4-2} = 5 - \frac{6}{-6} = 6$$

$$\frac{d^2y}{dx^2} = 10x + \frac{6}{(y-2)^2}\,\frac{dy}{dx}$$

$$= 10x + \frac{6}{(y-2)^2}\cdot\underbrace{\left(5x^2-\frac{6}{y-2}\right)}_{=6}$$

$$= 10x + \frac{36}{(y-2)^2}\Big|_{(-1,-4)} = -10 + \frac{36}{(-6)^2} = -10 + 1 = -9$$

$$\boxed{\frac{dy}{dx} = 6 \quad , \quad \frac{d^2y}{dx^2} = -9}$$

Work for problem 5(b)

to be tangent, $\frac{dy}{dx}$ would equal zero, and y would also be zero

$$\frac{dy}{dx} = 0 = 5x^2 - \frac{6}{y-2}$$

$$\frac{6}{y-2} = 5x^2$$

$$\frac{6}{-2} = 5x^2$$

$$x^2 = -\frac{3}{5}$$

since x^2 cannot be negative, there is no point possible such that the x-axis is tangent to the graph.

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5A2

Work for problem 5(c)

$P(x) = a_0 + a_1(x+1) + a_2(x+1)^2$

$P(-1) = a_0$

$P'(x) = a_1 + 2a_2(x+1)$

$P'(-1) = a_1$

$P''(x) = 2a_2$

$P''(-1) = 2a_2$

$f(x) = \sim$

$f(-1) = -4 \qquad a_0 = -4$

$f'(x) = 5x^2 - \frac{6}{y-2}$

$f'(-1) = 6 \qquad a_1 = 6$

$f''(x) = 10x + \frac{6}{(y-2)^2}\left[5x^2 - \frac{6}{y-2}\right]$

$f''(-1) = -9 \qquad 2a_2 = -9$

$a_2 = -4.5$

$P(x) = -4 + 6(x+1) - 4.5(x+1)^2$

Work for problem 5(d)

$y_1 = y_0 + f'(x_0, y_0)\Delta x \qquad (-1, -4)$

$y_1 = -4 + \left(5(-1)^2 - \frac{6}{-6}\right)(0.5)$

$= -4 + (5 + 1)(\frac{1}{2})$

$= -4 + 3 = -1$

$y_2 = y_1 + f'(x_1, y_1)\Delta x \qquad (-\frac{1}{2}, -1)$

$= -1 + \left(5(\frac{1}{4}) - \frac{6}{-3}\right)(\frac{1}{2})$

$= -1 + \left(\frac{5}{4} + 2\right)\frac{1}{2}$

$= -1 + \frac{13}{8} = \frac{5}{8}$

$f(0) \approx \frac{5}{8}$

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5B

Work for problem 5(a)

$$\left.\frac{dy}{dx}\right|_{(-1,-4)} = 5(-1)^2 - \frac{6}{-4-2} = 5+1=6$$

$$\frac{d^2y}{dx^2} = 10x - \left[\frac{-6\frac{dy}{dx}}{(y-2)^2}\right] = 10x + \frac{6\left(5x^2 - \frac{6}{y-2}\right)}{(y-2)^2}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(-1,-4)} = -10 + \frac{6(5+1)}{(-6)^2} = -10 + \frac{36}{36} = -9$$

Work for problem 5(b)

$$0 = \frac{dy}{dx}$$

$$5x^2 - \frac{6}{y-2} = 0$$

$$5x^2 = \frac{6}{y-2}$$

$$5(-1)^2 = \frac{6}{-4-2}$$

$$5 \neq -1$$

The graph can never be tangent to the x-axis because ~~that~~ would mean that the slope was 0. The slope of f(x) can never equal 0.

Continue problem 5 on page 13.

NO CALCULATOR ALLOWED

5B2

Work for problem 5(c)

$f(-1) = -4$

$f'(-1) = 6$

$f''(-1) = -9$

$f(x) = -4 + 6x - \frac{9x^2}{2!}$

Work for problem 5(d)

x	dx	$\frac{dy}{dx}$	y	dy
-1	0.5	6	-4	3
-0.5		$\frac{13}{4}$	-1	$\frac{13}{8}$
0			$\frac{5}{8}$	

$f(0) \approx \frac{5}{8}$

$\left.\frac{dy}{dx}\right|_{(-.5,-1)} = \frac{5}{4} - \frac{6}{-3}$

$= \frac{5}{4} + 2 = \frac{13}{4}$

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NO CALCULATOR ALLOWED

5C

Work for problem 5(a)

$\frac{dy}{dx} = 5x^2 - \frac{6}{y-2}$; $y = 2$ $y = f(x)$

$f(-1) = -4$

$\frac{dy}{dx} = 5(-1)^2 - \frac{6}{-4-2}$

$= 5 - \frac{6}{-6}$

$= 6$

$\frac{d^2y}{dx^2} = 10x + \frac{6}{(y-2)^2} \frac{dy}{dx}$

$= 10x + \frac{6}{(y-2)^2}\left[5x^2 - \frac{6}{y-2}\right]$

$= 10(-1) + \frac{6}{(-4-2)^2}[6] \Rightarrow -10 + \frac{6}{36} \cdot 6 \Rightarrow -10 + \frac{1}{6} \cdot 6 = 9$

Work for problem 5(b)

$0 = 5x^2 - \frac{6}{y-2}$

$\frac{6}{y-2} = 5x^2$

Yes, it's possible for the x-axis to be tangent to the graph of f at some point. This just means there is a horizontal tangent line and $\frac{dy}{dx} = 0$ somewhere. Where ever $\frac{6}{y-2} = 5x^2$, there is a horizontal tangent line for $f(x)$.

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NO CALCULATOR ALLOWED

5C2

Work for problem 5(c)

$f(-1) = -4$

$f'(-1) = 6$

$f''(-1) = 9$

$P_2(x) = -4 + 6(x-1) + \frac{9(x-1)^2}{2}$

Work for problem 5(d)

$\Delta x = .5$

x	y	$\Delta y = \frac{dy}{dx}\Delta x$	$\Delta y + y$
-1	-4	$= 6(.5)$	$3 + -4$
-.5	-1	$= \frac{9}{4}(.5)$	$\frac{9}{8} - 1$
0	$\frac{1}{8}$		

$f(0) \approx \frac{1}{8}$

$= 5x^2 - \frac{6}{y-2}$

$= \frac{5}{4} - \frac{-6}{-3}$

$= \frac{5}{4} + 2$

$= \frac{5+4}{4}$

$= \frac{9}{4} \times \frac{1}{2} \Rightarrow \frac{9}{8}$

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AP® CALCULUS BC
2006 SCORING COMMENTARY

Question 5

Overview

This problem presented students with a differential equation and asked questions about a particular solution satisfying a given initial condition. In part (a) they needed to evaluate the first and second derivatives at the initial condition, using implicit differentiation for the latter computation. In part (b) students could observe that $\frac{dy}{dx} > 0$ when $y = 0$ to help decide if it was possible for the x-axis to be tangent to the graph of the particular solution at some point. Part (c) asked for the second-degree Taylor polynomial at $x = -1$, which the students could compute using the initial condition and the results from part (a). In part (d) students needed to use Euler's method with two steps of equal size to approximate the value of the particular solution at $x = 0$. Because the differential equation was not separable, students were not expected to solve the equation in order to answer these questions about the behavior of the particular solution. All questions could be answered by working directly with the differential equation.

Sample: 5A
Score: 9

The student earned all 9 points.

Sample: 5B
Score: 6

This student earned 6 points: 3 points in part (a), 1 point in part (c), and 2 points in part (d). The work in part (a) is correct. In part (b) the student fails to let $y = 0$. Thus the student did not earn the first point and was unable to successfully answer and explain for the second point. In part (c) the student does not show the quadratic centered at $x = -1$; however, the coefficients are correct. Thus the student earned the second, but not the first, point. The work in part (d) is correct.

Sample: 5C
Score: 4

This student earned 4 points: 2 points in part (a), 1 point in part (c), and 1 point in part (d). In part (a) the student earned the first 2 points but did not earn the last point due to a computational error. In part (b) the student never considers $y = 0$ so could not earn the first point and was therefore ineligible for the last point. In part (c) the quadratic polynomial is centered at $x = 1$ instead of at $x = -1$, so the student did not earn the first point. The coefficients are correct and/or consistent with part (a) and the second point was earned. In part (d) the student uses two steps of Euler's method, which earned the first point. A computational error prevents the student from getting the correct approximation to $f(0)$.